Assignment 6 - MATC46

Due March 31 2016

Question 1 A 3×3 square plate with $\alpha = 1/2$ is heated in such a way that the temperature is distributed with $f(x, y) = y$. After that the temperature at its left and right edges being held at 0 and the lower and upper edges being insulated. Find a series expansion that gives the temperature in the plate for $t > 0$.

Solution The heat equation dictates

$$u_t = \alpha^2 \Delta u \implies 4u_t = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Assume the solution is separable to deduce (i.e. $u(x, y, t) = X(x)Y(y)T(t)$),

$$4\frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2 \quad \text{where } \lambda \in \mathbb{R}$$

so we see

$$T(t) = Ae^{-\lambda^2 t/4}$$

since heat is leaving the system (this forces a negative eigenvalue). Splitting the spacial part shows

$$X'' = -m^2 X \quad \& \quad Y'' = -n^2 Y \quad \text{where} \quad m^2 + n^2 = \lambda^2$$

$$\implies X = A\cos mx + B\sin mx \quad \& \quad Y = C\cos ny + D\sin ny$$

Define our space to be $\Omega = [0, 3] \times [0, 3]$ and we see the boundary data for X gives:

$$X(0) = X(3) = 0 \implies A = 0 \quad \& \quad \sin 3m = 0 \implies m = \frac{n_1\pi}{3}, \quad n_1 \in \mathbb{N}$$

For y, we see

$$Y'(0) = Y'(3) = 0 \implies D = 0 \quad \& \quad \sin 3n = 0 \implies n = \frac{n_2\pi}{3}, \quad n_2 \in \mathbb{N}$$

Now using linearity, we know the solution must be a sum of all the eigenfunctions we've found:

$$u(x, y, t) = \sum_{n_1, n_2 \geqslant 0} c_{n_1,n_2} \exp\left(-\frac{\pi^2(n_1^2 + n_2^2)}{36}t\right) \sin\left(\frac{n_1\pi x}{3}\right) \cos\left(\frac{n_2\pi y}{3}\right)$$

The initial data will give us the coefficients of the series by using orthogonality of the eigenfunctions. We see $f(x, y) = y$ at $t = 0$, so when $n_2 \neq 0$,we have

$$c_{n_1,n_2} = \frac{4}{9}\int_0^3 \int_0^3 y \sin(n_1\pi x/3)\cos(n_2\pi y/3)dxdy = -12\frac{(1 - (-1)^{n_1})(1 - (-1)^{n_2})}{\pi^3 n_1 n_2^2}$$

and when $n_2 = 0$

$$c_{n_1,0} = \frac{2}{9}\int_0^3 \int_0^3 y \sin(n_1\pi x/3)dxdy = 3\frac{1 - (-1)^{n_1}}{\pi n_1}$$

□

Question 2 A circular plate with radius 2 and with $\alpha = 1/2$ is heated in such a way that the temperature is distributed with $f(r,\theta) = 2 - r$. After that the temperature on the boundary being held at 0. Write the formal Bessel-Fourier expansion that gives the temperature in the plate for $t > 0$. Write the coefficients via integrals involving Bessel functions.

Solution The heat equation dictates

$$4u_t = \Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Note that the initial data is circularly symmetric, thus u is independent of θ. So we assume the solution is separable, i.e. $u = R(r)T(t)$, then we obtain

$$2\frac{T'}{T} = \frac{1}{rR}\frac{\partial}{\partial r}(rR') = -\lambda^2, \quad \lambda \in \mathbb{R}$$

so we see

$$T(t) = Ae^{-\lambda^2 t/4}$$

since heat is leaving the system. Let $R(r) = J(\lambda r)$ so the ODE becomes

$$r^2 J'' + rJ' + r^2 J = 0$$

The solutions are given by Bessel functions of 0th order(as we've seen in class and we'll them as J_0) since this is Bessel's equation. The boundary constraint enforces that

$$J_0(2\lambda) = 0 \implies \lambda = \frac{\lambda_n}{2} \quad \text{is the the } n\text{-th zero of } J_0$$

(since one may show $J_0(x)$ has infinitely many zero's). Thus (by linearity)

$$u(r,\theta,t) = \sum_{n\geqslant 1} c_n e^{-\lambda_n^2 t/16} J_0(\lambda_n r)$$

We know that these Bessel functions are orthogonal via

$$\frac{1}{2}\int_0^2 rJ_0\left(\frac{\lambda_{n_1}}{2}r\right) J_0\left(\frac{\lambda_{n_2}}{2}r\right) dr = \left(J_0'(\lambda_{n_1})\right)^2 \delta_{n_1,n_2}$$

Thus using Fourier's trick (multiplying u by $rJ_0(\lambda_n)$ and integrating at $t = 0$), we see

$$c_n = \frac{1}{(J_0'(\lambda_n))^2}\int_0^2 r(2-r)J_0\left(\frac{\lambda_n}{2}r\right) dr$$

□

Question 3 Find the displacement $u(r,\theta)$ of a circular membrane of radius 1 with $a^2 = 4$ clamped along its circumference if its initial displacement is

$$u(r,\theta,0) = J_0(\lambda_1 r) - 0.25 J_0(\lambda_3 r).$$

and $u_t(0,r,\theta) = 0$. Here J_0 is the Bessel function of first kind of order 0 and λ_k are it's zeros.

Solution The wave equation dictates

$$u_{tt} = a^2 \Delta u = 4\Delta u = \frac{4}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{4}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Note that the initial data is circularly symmetric, thus u is independent of θ. So we assume the solution is separable, i.e. $u = R(r)T(t)$, we see

$$\frac{T''}{4T} = \frac{1}{rR}\frac{\partial}{\partial r}(rR') = -\lambda^2, \quad \lambda \in \mathbb{R}$$

Which shows us that T has solutions of the form (due to the initial data)

$$T(t) = A\sin 2\lambda t + B\cos 2\lambda t \quad \& \quad T'(0) = 0 \implies T(t) = B\cos 2\lambda t$$

and

$$\frac{r}{R}\frac{\partial}{\partial r}(rR') + \lambda^2 r^2 = 0$$

$R(r) = J(\lambda r)$, then the ODE becomes

$$r^2 J'' + rJ' + r^2 J = 0$$

which is Bessel's Equation again, and we know the solutions are given by the Bessel function of order 0, J_0. The boundary is clamped, i.e. u=0, thus

$$J_0(\lambda) = 0 \implies \lambda = \lambda_n \quad \text{is the the } n\text{-th zero of } J_0$$

Therefore, by linearity we have

$$u(r,\theta,t) = \sum_{n\geqslant 1} c_n J_0(\lambda_n r)\cos(2\lambda_n t)$$

The initial data shows us by comparison we have

$$u(r,\theta,t) = J_0(\lambda_1 r)\cos(2\lambda_1 t) - 0.25 J_0(\lambda_3 r)\cos(2\lambda_3 t)$$

by orthogonality of the Bessel functions. □

Question 4 Find the steady state temperature $u(r,\phi)$ in a sphere of unit radius if the temperature is independent of the polar angle θ and satisfies the boundary condition

$$u(1,\phi) = P_1(\cos\phi) - P_3(\cos\phi).$$

Here P_n is the n-th Legendre polynomial.

Solution Notice the steady state temperature must satisfy $\Delta u = 0$ in the sphere, and

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \underbrace{\cancel{\frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}}}_{=0} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)$$

So if we assume the solution is separable, $u = R(r)\Phi(\phi)$, we see that

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = -\frac{1}{\Phi\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial\Phi}{\partial\phi}\right) = \lambda \in \mathbb{R}$$

since the LHS is independent of ϕ and the RHS is independent of r, they both must be equal to a constant. Thus

$$r^2R'' + 2rR' - \lambda R = 0 \quad \& \quad \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial\Phi}{\partial\phi}\right) + \lambda\Phi = 0$$

Regularity at the North and South poles of the sphere forces $\lambda_n = n(n+1)$, and we know

$$\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}(P_n(\cos\phi))\right) + n(n+1)P_n(\cos\phi) = 0$$

where P_n is the n-th Legendre polynomial which satisfies

$$\int_0^\pi P_m(\cos\phi)P_n(\cos\phi)\sin\phi\, d\phi = \frac{2}{2n+1}\delta_{m,n}$$

We see the ODE in r is an Euler equation, we test $R = r^k$ and find

$$k(k+1) = n(n+1) \implies k = n \quad \& \quad k = -n-1$$

but regularity at the origin forces $R(r) = Ar^n$. Therefore, by linearity we have

$$u(r,\phi) = \sum_{n\geqslant 0} a_n r^n P_n(\cos\phi)$$

The boundary data shows

$$P_1(\cos\phi) - P_3(\cos\phi) = \sum_{n\geqslant 0} a_n P_n(\cos\phi) \implies a_1 = 1, a_3 = -1 \quad \text{and} \quad a_{else} = 0$$

Thus the solution is given by

$$u(r,\phi) = rP_1(\cos\phi) - r^3P_3(\cos\phi)$$

□

Question 5 Consider the flow of heat in an infinitely long cylinder of radius 1 with $\alpha = 1/5$. Let the surface of the cylinder temperature be held at 0 and let the initial distribution of the temperature be

$$u(r,\theta,z)|_{t=0} = 4J_0(\lambda_2 r) - 0.1J_0(\lambda_5 r)$$

Solution The heat equation dictates

$$25u_t = \Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2}$$

Assume the solution is separable i.e. $u(r,\theta,z,t) = R(r)T(t)$, (where Z and Θ is dropped is the problem is independent of z and θ) then we obtain

$$5\frac{T'}{T} = \frac{1}{rR}\frac{\partial}{\partial r}(rR') = -\lambda^2$$

so we see

$$T(t) = A\exp(-\lambda^2 t/25)$$

since heat is leaving the system and

$$\frac{r}{R}\frac{\partial}{\partial r}(rR') + \lambda^2 r^2 = 0$$

Let $R(r) = J(\lambda r)$ then we see

$$r^2 J'' + rJ' + r^2 J = 0$$

which has solutions of $J_0(\lambda r)$, and the boundary data implies

$$J_0(\lambda) = 0 \implies \lambda = \lambda_n \quad \text{is the the } n\text{-th zero of } J_0$$

Therefore

$$u(r,t) = \sum_{n \geqslant 1} c_n J_0(\lambda_n r) \exp(-\lambda_n^2 t/25)$$

By orthogonality(see previous questions), we see the initial data implies

$$u(r,t) = 4J_0(\lambda_2 r) \exp(-\lambda_2^2 t/25) - 0.1 J_0(\lambda_5 r) \exp(-\lambda_2^2 t/25)$$

□