Assignment 9

MATC34 – Complex Variables – Fall 2015

Solutions

Question 1 Evaluate the integral using residues

$$I_1 = \int_0^\infty \frac{t^2 dt}{t^4+1}$$

Solution Consider

$$f(z) = \frac{z^2}{z^4+1}$$

evaluated along the half circle of radius R with positive orientation, i.e.

$$\gamma = \underbrace{\{z : |z| = R, \operatorname{Arg} z \in (0,\pi)\}}_{=\gamma_\theta} \bigcup \underbrace{\{z = x+iy : y = 0, |x| \leqslant R\}}_{=\gamma_x}$$

Thus we see as long as $R > 1$, we have two roots of z^4+1 living inside the contour, specifically $z = \frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}$, and they're obviously simple. Therefore the residue theorem allows us to conclude

$$\int_\gamma f(z)dz = \int_{\gamma_x} f(z)dz + \int_{\gamma_\theta} f(z)dz = 2\pi i \left[\operatorname{Res}\left(f, \frac{1+i}{\sqrt{2}}\right) + \operatorname{Res}\left(f, \frac{-1+i}{\sqrt{2}}\right)\right]$$

Jordan's lemma with an obvious bound of $M_R = const/R$ gives us

$$\lim_{R\to\infty} \int_{\gamma_\theta} f(z)dz = 0$$

If we write out the other component, we see

$$\lim_{R\to\infty} \int_{\gamma_x} f(z)dz = 2I_1$$

Thus

$$\begin{aligned}
I_1 =& \pi i \left[\operatorname{Res}\left(f, \frac{1+i}{\sqrt{2}}\right) + \operatorname{Res}\left(f, \frac{-1+i}{\sqrt{2}}\right)\right] \\
=& \pi i \left[\lim_{z\to\frac{1+i}{\sqrt{2}}} \left(z - \frac{1+i}{\sqrt{2}}\right) \frac{z^2}{z^4+1} + \lim_{z\to\frac{-1+i}{\sqrt{2}}} \left(z - \frac{-1+i}{\sqrt{2}}\right) \frac{z^2}{z^4+1}\right] \\
=& \pi i \left[\lim_{z\to\frac{1+i}{\sqrt{2}}} \frac{1}{4z} + \lim_{z\to\frac{-1+i}{\sqrt{2}}} \frac{1}{4z}\right] \\
=& \pi i \left[\frac{1-i}{4\sqrt{2}} - \frac{1+i}{4\sqrt{2}}\right] \\
=& \frac{\pi}{2\sqrt{2}}
\end{aligned}$$

□

Question 2 Evaluate the integral using residues

$$I_2 = \int_{-\infty}^{\infty} \frac{\cos(x)dx}{(x^2+9)(x^2+4)}$$

Solution Consider

$$f(z) = \frac{e^{iz}}{(z^2+9)(z^2+4)}$$

evaluated along the half circle of radius R with positive orientation, i.e.

$$\gamma = \underbrace{\{z : |z| = R, \mathrm{Arg} z \in (0,\pi)\}}_{=\gamma_\theta} \bigcup \underbrace{\{z = x+iy : y = 0, |x| \leqslant R\}}_{=\gamma_x}$$

We see as long as $R > 3$, we have that $z = 3i, 2i$ are simple poles of $f(z)$. Therefore

$$\int_\gamma f(z)dz = \int_{\gamma_\theta} f(z)dz + \int_{\gamma_x} f(z)dz = 2\pi i(\mathrm{Res}(f,3i) + \mathrm{Res}(f,2i))$$

Jordan's lemma with the obvious bound of $M_R = const/R^3$ gives us

$$\lim_{R\to\infty} \int_{\gamma_\theta} f(z)dz = 0$$

The other component can be written as

$$\lim_{R\to\infty} \int_{\gamma_x} f(z)dz = \int_{-\infty}^{\infty} \frac{\cos(x)dx}{(x^2+9)(x^2+4)} + i\int_{-\infty}^{\infty} \frac{\sin(x)dx}{(x^2+9)(x^2+4)} = I_2 + i\int_{-\infty}^{\infty} \frac{\sin(x)dx}{(x^2+9)(x^2+4)}$$

Thus we simply need the real component of the residues to compute the integral, we see

$$\begin{aligned}
I_2 =& \Re\left[2\pi i\left(\mathrm{Res}(f,3i) + \mathrm{Res}(f,2i)\right)\right] \\
=& \Re\left[2\pi i\left(\lim_{z\to 3i}(z-3i)\frac{e^{iz}}{(z+3i)(z-3i)(z-2i)(z+2i)} + \lim_{z\to 2i}(z-2i)\frac{e^{iz}}{(z+3i)(z-3i)(z-2i)(z+2i)}\right)\right] \\
=& \Re\left[2\pi i\left(\frac{e^{-3}}{6i\times i\times 5i} + \frac{e^{-2}}{5i\times -i\times 4i}\right)\right] \\
=& \pi\left(\frac{e^{-2}}{10} - \frac{e^{-3}}{15}\right)
\end{aligned}$$

□

Question 3 Evaluate the integral using residues

$$I_3 = \int_0^{\infty} \frac{x\sin(ax)}{x^2+b^2}dx, \quad a > 0$$

Solution Consider

$$f(z) = \frac{ze^{iaz}}{z^2+b^2}$$

evaluated along the half circle of radius R with positive orientation, i.e.

$$\gamma = \underbrace{\{z : |z| = R, \mathrm{Arg} z \in (0,\pi)\}}_{=\gamma_\theta} \bigcup \underbrace{\{z = x+iy : y = 0, |x| \leqslant R\}}_{=\gamma_x}$$

We see as long as $R > |b|$ we have that $z = i|b|$ ($|b| > 0$) is a simple pole of $f(z)$ inside the contour. Thus

$$\int_{\gamma} f(z)dz = \int_{\gamma_x} f(z)dz + \int_{\gamma_\theta} f(z)dz = 2\pi i \text{Res}(f, i|b|)$$

Again we apply Jordan's lemma with the obvious bound $M_R = e^{-aR\sin(\theta)}$, we see that

$$\lim_{R\to\infty} \int_{\gamma_\theta} f(z)dz = 0$$

Writing out the other component, we see

$$\int_{\gamma_x} f(z)dz = \underbrace{\int_{-R}^{R} \frac{x\cos(ax)}{x^2+b^2}dx}_{odd} + i\underbrace{\int_{-R}^{R} \frac{x\sin(ax)}{x^2+b^2}dx}_{even} = 2i\int_0^R \frac{x\sin(ax)}{x^2+b^2}dx \to 2iI_3$$

in the limit as $R \to \infty$. Thus we see

$$I_3 = \pi \text{Res}(f, i|b|) = \pi \lim_{z\to i|b|} (z - i|b|)\frac{ze^{iaz}}{z^2+b^2} = \frac{\pi e^{-a|b|}}{2} \qquad |b| > 0$$

When $b = 0$, we see that it's an integral we've already computed. Namely

$$\int_0^\infty \frac{\sin(ax)}{x}dx = \int_0^\infty \frac{\sin(y)}{y}dy = \frac{\pi}{2}, \quad b = 0$$

Thus we conclude

$$\int_0^\infty \frac{x\sin(ax)}{x^2+b^2}dx = \frac{\pi e^{-a|b|}}{2} \qquad a > 0, \quad b \in \mathbb{R}$$

□

Question 4 Evaluate the integral using residues

$$I_4 = \int_0^\infty \frac{\sin(x)}{x(x^2+1)}dx$$

Solution Consider

$$f(z) = \frac{e^{iz}}{z(z^2+1)}$$

evaluated along the half circle of radius R with a small upward arc of radius ϵ avoiding the issue at $0z = 0$ (with positive orientation of course) i.e.

$$\gamma = \underbrace{\{z : |z| = R, \text{Arg}z \in (0,\pi)\}}_{=\gamma_\theta} \bigcup \underbrace{\{z = x + iy : y = 0, \epsilon < |x| \leqslant R\}}_{=\gamma_x} \bigcup \underbrace{\{z : |z| = \epsilon, \text{Arg}z \in (0,\pi)\}}_{=\gamma_\epsilon}$$

As long as $R > 1 > \epsilon$, we'll have the simple pole $z = i$ inside the contour, thus

$$\int_{\gamma} f(z)dz = \int_{\gamma_\theta} f(z)dz + \int_{\gamma_x} f(z)dz + \int_{\gamma_\epsilon} f(z)dz = 2\pi i \text{Res}(f, i)$$

Jordan's lemma with a simple bound like $M_R = const/R^2$ tells us that

$$\lim_{R\to\infty} \int_{\gamma_\theta} f(z)dz = 0$$

We see taking the limit of $\epsilon \to 0$ doesn't affect the other two components much since

$$\int_{\gamma_\epsilon} f(z)dz = \int_\pi^0 f(\epsilon e^{i\theta})\epsilon ie^{i\theta}d\theta = \int_\pi^0 \frac{e^{i\epsilon e^{i\theta}}}{\epsilon^2 e^{2i\theta}+1} id\theta \to \int_\pi^0 id\theta = -\pi i$$

in the limit as $\epsilon \to 0$. Now we simply have to take the imaginary part of the integral on the x axis since

$$\lim_{\epsilon\to 0} \Im \int_{\gamma_x} f(z)dz = \underbrace{\int_{-R}^{R} \frac{\sin(x)}{x(x^2+1)}dx}_{even} \to 2I_4$$

in the limit as $R \to \infty$. Now if we put everything together, we see that

$$I_4 = \pi\left(\frac{1}{2} + \text{Res}(f,i)\right) = \pi\left(\frac{1}{2} + \lim_{z\to i}(z-i)\frac{e^{iz}}{z(z^2+1)}\right) = \pi\left(\frac{1-e^{-1}}{2}\right)$$

□

Alternate Solution Use a partial fraction decomposition on the integral to obtain

$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)}dx = \int_0^\infty \frac{\sin(x)}{x}dx - \int_0^\infty \frac{x\sin(x)}{x^2+1}dx$$

Now use the general formula from the previous question, i.e.

$$\int_0^\infty \frac{x\sin(ax)}{x^2+b^2}dx = \frac{\pi e^{-a|b|}}{2} \quad a>0, \quad b\in\mathbb{R}$$

Take $a=1$ then $b=0$ for the first and $b=1$ for the second, we see

$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)}dx = \pi\left(\frac{1-e^{-1}}{2}\right)$$

□

Question 5 Evaluate the integral using residues

$$I_5 = \int_0^\pi \frac{dt}{10+8\cos(t)}$$

Solution Notice the integrand can be extended to

$$\int_0^\pi \frac{dt}{10+8\cos(t)} = \frac{1}{2}\int_0^{2\pi} \frac{dt}{10+8\cos(t)}$$

Now let's change back to z coordinates using $z=e^{it}$, $\gamma = \{z : |z| = 1\}$, we see

$$\cos(t) = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}, \quad \frac{-i}{z}dz = dt$$

Thus

$$\int_0^\pi \frac{dt}{10+8\cos(t)} = -\frac{i}{4}\int_\gamma \underbrace{\frac{dz}{5z+2(z^2+1)}}_{f(z)dz}$$

We check the roots of the denominator,

$$2z^2+5z+2=0 \implies z = \frac{-5\pm\sqrt{25-2^4}}{4} = \frac{-5\pm 3}{4}$$

Thus the only pole inside the domain is $z=-1/2$, and it is simple. Therefore

$$\int_0^\pi \frac{dt}{10+8\cos(t)} = -\frac{i}{4}\times 2\pi i \text{Res}(f,-1/2) = \frac{\pi}{4}\lim_{z\to -1/2}(z+1/2)\frac{1}{(z+8/4)(z+1/2)} = \frac{\pi}{6}$$

□