A4-Q1

MATC34 – Complex Variables – Fall 2015

Solutions

Q1 Verify Cauchy's Theorem (without using it) for $f(z) = 1/z$ around $\{z : |z| = 4\}$ and $\{z : ||z||_\infty = 1\}$. i.e. Compute

$$\int_{\{z:|z|=4\}} f(z)dz \quad \& \quad \int_{\{z:||z||_\infty=1\}} f(z)dz$$

and show they are equal.

Solution You have to parametrize both curves, so let's first recall the definition of how to compute line integrals.

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Now let's start with the circle. The simplest parameterization you know is through Euler's Identity i.e.

$$z = 4e^{i\theta} = 4(\cos\theta + i\sin\theta)$$

i.e. our the parametrization is given by $\gamma(t) = 4e^{it}$, $\gamma'(t) = 4ie^{it}$, and we run $t \in [0, 2\pi)$. Thus

$$\boxed{\int_{\{z:|z|=4\}} \frac{dz}{z} = \int_0^{2\pi} \frac{4ie^{it}}{4e^{it}}dt = i\int_0^{2\pi} dt = 2\pi i}$$

To handle the square, let's break the integral into 4 pieces (i.e. the 4 sides) and be mindful of the orientation: the right side is given by the parameterization $\gamma_1(t) = 1 + it$ with t going from -1 to 1, the left side is given by $\gamma_2(t) = -1 + it$ with t going from 1 to -1, the top side is given by $\gamma_3(t) = t + i$ with t going from 1 to -1, and the bottom side is given by $\gamma_4(t) = t - i$ with t going from -1 to 1. Thus we have

$$\begin{aligned}
\int_{\{z:||z||_\infty=1\}} \frac{dz}{z} &= \int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z} + \int_{\gamma_3} \frac{dz}{z} + \int_{\gamma_4} \frac{dz}{z} \\
&= \int_{-1}^{1} \frac{idt}{1+it} + \int_1^{-1} \frac{idt}{-1+it} + \int_1^{-1} \frac{dt}{t+i} + \int_{-1}^{1} \frac{dt}{t-i} \\
&= \int_{-1}^{1} \frac{(i+t)dt}{1+t^2} + \int_1^{-1} \frac{(-i+t)dt}{1+t^2} + \int_1^{-1} \frac{(t-i)dt}{1+t^2} + \int_{-1}^{1} \frac{(t+i)dt}{1+t^2}
\end{aligned}$$

Notice that all components with t in the numerator are odd, thus

$$\int_{\{z:||z||_\infty=1\}} \frac{dz}{z} = 4i\int_{-1}^{1} \frac{dt}{1+t^2}$$

You should be able to handle the integral from your knowledge from a standard calculus class(use a trig substitution), thus we see

$$\boxed{\int_{\{z:||z||_\infty=1\}} \frac{dz}{z} = 4i\int_{-1}^{1} \frac{dt}{1+t^2} = 4i\frac{\pi}{2} = 2\pi i}$$

Notice that the expression is <u>not multi-valued or anything weird</u>... Clearly the integrals are the same value.

Alternate integration Instead of rationalizing the denominator, let's handle the logarithm we'd get otherwise.

$$\begin{aligned}\int_{\{z:||z||_\infty=1\}} \frac{dz}{z} &= \int_{-1}^{1} \frac{idt}{1+it} + \int_{1}^{-1} \frac{idt}{-1+it} + \int_{1}^{-1} \frac{dt}{t+i} + \int_{-1}^{1} \frac{dt}{t-i} \\ &= \ln(1+it)\Big|_{-1}^{1} - \ln(-1+it)\Big|_{-1}^{1} - \ln(t+i)\Big|_{-1}^{1} + \ln(t-i)\Big|_{-1}^{1}\end{aligned}$$

Note that

$$1 \pm i = \sqrt{2}e^{\pm i\pi/4}, \quad -1 \pm i = \sqrt{2}e^{i\pi \pm i\pi/4}, \quad \pm 1 + i = \sqrt{2}e^{i\pi/2 \pm i\pi/4}, \quad \pm 1 - i = \sqrt{2}e^{3i\pi/2 \pm i\pi/4}$$

Thus, we have (note that whatever branch you choose, this won't change the result since it gets added and subtracted)

$$\begin{aligned}\ln(1+it)\Big|_{-1}^{1} &= \ln(1+i) - \ln(1-i) = i\frac{\pi}{2} \\ -\ln(-1+it)\Big|_{-1}^{1} &= \ln(-1-i) - \ln(-1+i) = i\frac{\pi}{2} \\ -\ln(t+i)\Big|_{-1}^{1} &= \ln(-1+i) - \ln(1+i) = i\frac{\pi}{2} \\ \ln(t-i)\Big|_{-1}^{1} &= \ln(1-i) - \ln(-1-i) = i\frac{\pi}{2}\end{aligned}$$

If we put it all together, we see

$$\boxed{\int_{\{z:||z||_\infty=1\}} \frac{dz}{z} = 4\left(i\frac{\pi}{2}\right) = 2\pi i}$$