Tutorial 9

MAT334 – Complex Variables – Spring 2016

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SOLUTIONS

2.6 - # 3 Compute

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, \quad a, b > 0$$

Solution Define the polynomials

$$P(z) = 1 \quad \& \quad Q(z) = (z^2+a^2)(z^2+b^2)$$

and notice Q has zeros in the upper half-plane at $z = ai, bi$. Now

$$\frac{P(z)}{Q(z)} = \frac{1}{(x-ai)(x+ai)(x+bi)(x-bi)}$$

Consider the contour of a semi-circle of radius R with base on the x-axis. We see the residue at the two poles are given by

$$\text{Res}\left(\frac{P}{Q}; ai\right) = \frac{1}{(ai+ai)(ai+bi)(ai-bi)} = \frac{1}{2ia(b^2-a^2)}$$

$$\text{Res}\left(\frac{P}{Q}; bi\right) = \frac{1}{(bi+bi)(bi-ai)(bi+ai)} = -\frac{1}{2ib(b^2-a^2)}$$

Taking the limit as $R \to \infty$ of the contour with the residue theorem tells us

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i \left[\text{Res}\left(\frac{P}{Q}; ai\right) + \text{Res}\left(\frac{P}{Q}; bi\right)\right] = \frac{\pi}{ab(a+b)}$$

$\square$

2.6 - # 7 Compute

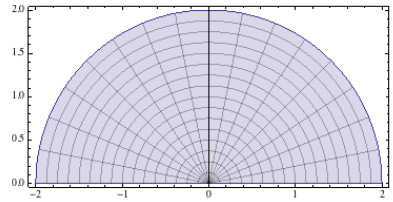
$$\int_{-\infty}^{\infty} \frac{\cos x}{(x+\alpha)^2+\beta^2} dx$$

Solution Consider

$$f(z) = \frac{e^{iz}}{(z+\alpha)^2+\beta^2} = \frac{e^{iz}}{(z-(-\alpha+i\beta))(z-(-\alpha-i\beta))}$$

with a contour of a semi-circle of radius R with base along on the x-axis.

$$\gamma = \underbrace{\{z : |z| = R, \arg z \in (0,\pi)\}}_{\gamma_R} \cup \underbrace{\{z = x+iy : -R < x < R, y = 0\}}_{\gamma_x}$$



We see a simple pole in the contour at $z = -\alpha + i\beta$ (assuming that $\beta > 0$ with no loss of generalities). Notice on γ_R, we see

$$\left|\int_{\gamma_R} f(z)dz\right| \leqslant \frac{Const}{R}$$

(Noting $|e^{iz}| \leqslant e^{-R\sin\theta}$) Thus in the limit as $R \to \infty$, it will not contribute. Using the residue theorem, we conclude

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x+\alpha)^2+\beta^2} dx = \Re\left[2\pi i \operatorname{Res}\left(f(z) : -\alpha + i\beta\right)\right] = \Re\left[2\pi i \frac{e^{i(-\alpha+i\beta)}}{2i\beta}\right] = \frac{\pi e^{-\beta}\cos(\alpha)}{\beta}$$

□

2.6 - # 9 Compute

$$\int_0^{2\pi} \frac{d\theta}{(2-\sin\theta)^2}$$

Solution Define $\gamma = \{z = e^{i\theta} : |z| = 1\}$, thus

$$\sin\theta = \frac{z^2-1}{2iz} \quad \& \quad d\theta = \frac{dz}{iz}$$

we see

$$\int_0^{2\pi} \frac{d\theta}{(2-\sin\theta)^2} = 4i \int_\gamma \frac{zdz}{(4iz - z^2 + 1)^2}$$

The poles of the denominator are given by

$$z_{pole} = \frac{4i \pm \sqrt{-16+4}}{2} = (2 \pm \sqrt{3})i$$

and $(2-\sqrt{3})i \in \{z : |z| < 1\}$. Thus the Residue Theorem gives us

$$\int_0^{2\pi} \frac{d\theta}{(2-\sin\theta)^2} = 4i * 2\pi i \operatorname{Res}\left(f; (2-\sqrt{3})i\right)$$

where

$$f(z) = \frac{z}{(z-(2+\sqrt{3})i)^2(z-(2-\sqrt{3})i)^2}$$

Note we may compute the residue of a pole order order n by

$$\operatorname{Res}(f; z_0) = \lim_{z\to z_0} \frac{1}{(n-1)!} \frac{d^{(n-1)}}{dz^{(n-1)}} (z-z_0)^n f(z)$$

Thus our pole of order 2 has a residue of

$$\operatorname{Res}\left(f; (2-\sqrt{3})i\right) = \lim_{z\to(2-\sqrt{3})i} \frac{d}{dz} \frac{z}{(z-(2+\sqrt{3})i)^2} = \lim_{z\to(2-\sqrt{3})i} \frac{(z-(2+\sqrt{3})i) - 2z}{(z-(2+\sqrt{3})i)^3} = -\frac{1}{6\sqrt{3}}$$

Plugging the residue into the conclusion of the residue theorem reveals

$$\int_0^{2\pi} \frac{d\theta}{(2-\sin\theta)^2} = \frac{4\pi}{3\sqrt{3}}$$

2.6 - # 13 Compute the integral using a "keyhole" contour

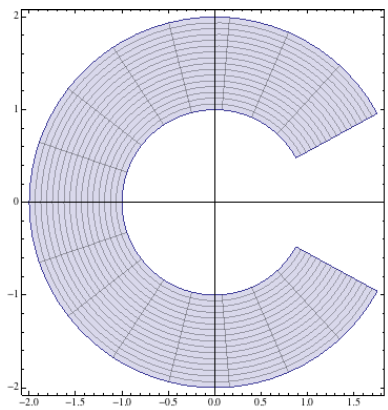
$$I = \int_0^\infty \frac{x^\alpha}{x^2+3x+2}dx, \quad \alpha \in (0,1)$$

Solution Consider

$$f(z) = \frac{z^\alpha}{z^2+3z+2} = \frac{\exp(\alpha \ln|z| + \alpha \arg(z))}{(z+2)(z+1)}$$

on $(R > r > 0)$

$$\gamma = \underbrace{\{z = Re^{i\theta} : \theta \in (\epsilon, 2\pi - \epsilon)\}}_{\gamma_R} \cup \underbrace{\{z = re^{i\theta} : \theta \in (\epsilon, 2\pi - \epsilon)\}}_{\gamma_r} \cup \underbrace{\{z = te^{i\epsilon} : t \in [r, R]\}}_{\gamma_{\epsilon-top}} \cup \underbrace{\{z = te^{i(2\pi-\epsilon)} : t \in [r, R]\}}_{\gamma_{\epsilon-bot}}$$



with the CCW orientation. Notice that -2 and -1 (the simple poles) live inside the contour, so we may apply the residue theorem as long at the other components are well behaved. We see

$$\left|\int_{\gamma_R} f(z)dz\right| \leqslant \frac{Const}{R^{1-\alpha}}$$

which will die in the limit as $R \to \infty$ as long as $\alpha < 1$. We also see

$$\left|\int_{\gamma_r} f(z)dz\right| \leqslant Const * r^{1+\alpha}$$

which will die in the limit as $r \to 0$ when $\alpha > 0$. Notice that

$$\int_{\gamma_{\epsilon-top}} f(z)dz \to I \quad \& \quad \int_{\gamma_{\epsilon-bot}} fdz \to -e^{2\pi\alpha i}I$$

in the limit as $\epsilon, r \to 0$ and $R \to \infty$ (- sign due to the reversed orientation). The residues are calculated to be

$$\text{Res}(f : -2) = \frac{\exp(\alpha \log|2| + i\alpha\pi)}{-1} = -2^\alpha e^{i\alpha\pi}$$

$$\text{Res}(f : -1) = \frac{\exp(\alpha \log|1| + i\alpha\pi)}{1} = e^{i\alpha\pi}$$

Thus the residue theorem gives us (in conjunction with the limits of R, r, ϵ)

$$(1 - e^{2\pi\alpha i})\int_0^\infty \frac{x^\alpha}{x^2+3x+2}dx = 2\pi i\left(\text{Res}(f : -2) + \text{Res}(f : -1)\right) = 2\pi i e^{i\alpha\pi}(1 + 2^\alpha)$$

$$\implies \int_0^\infty \frac{x^\alpha}{x^2+3x+2}dx = \frac{2\pi i e^{i\alpha\pi}(1-2^\alpha)}{1-e^{2\pi\alpha i}} = \pi(2^\alpha - 1)\frac{2i}{e^{\alpha\pi i} - e^{-\alpha\pi i}} = \frac{\pi(2^\alpha - 1)}{\sin(\alpha\pi)}$$

□

2.6 - # 17 Compute

$$I = \int_0^\infty \frac{\log x}{1+x^2} dx$$

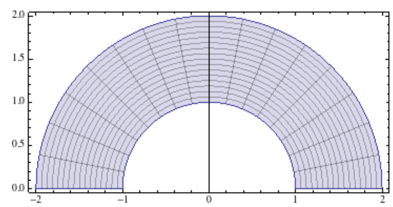
(note that the integral is improper since $\log 0$ isn't defined.)

Solution Take the plane with the negative imaginary axis removed and define

$$f(z) = \frac{\log z}{(z-i)(z+i)}, \quad z \in \mathbb{C} \setminus \{z = iy : y \leqslant 0\}$$

where log takes imaginary values from $(-\pi/2, 3\pi/2)$. Notice the pole at $z = i$. Take the the contour

$$\gamma = \underbrace{\{z = Re^{i\theta} : \theta \in (0,\pi)\}}_{\gamma_R} \cup \underbrace{\{z = re^{i\theta} : \theta \in (0,\pi)\}}_{\gamma_r} \cup \underbrace{\{z = x : R > |x| > r\}}_{\gamma_x}$$



with the CCW orientation, then i is in the contour. Note we have control of all contour pieces by

$$\left| \int_{\gamma_R} f dz \right| \leqslant Const * \frac{\log R}{R}$$

$$\left| \int_{\gamma_r} f dz \right| \leqslant Const * r * \log r$$

and

$$\int_{\gamma_x} f(x) dx = \int_{-R}^{-r} \frac{\log x}{1+x^2} dx + \int_r^R \frac{\log x}{1+x^2} dx = 2 \int_r^R \frac{\log x}{1+x^2} dx + i\pi \int_r^R \frac{dx}{1+x^2}$$

The residue at i is given by

$$\text{Res}\,(f(z); i) = \frac{\log(i)}{2i} = \frac{\pi}{4}$$

Now the residue theorem gives us

$$\int_\gamma f(z) dz = 2 \int_0^\infty \frac{\log x}{1+x^2} dx + i\pi \int_0^\infty \frac{dx}{1+x^2} = 2\pi i \times \frac{\pi}{4} = 0 + i\frac{\pi^2}{2}$$

$$\implies \int_0^\infty \frac{\log x}{1+x^2} dx = 0 \quad \& \quad \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

□