Tutorial 5

MAT334 – Complex Variables – Spring 2016

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Solutions

2.1 - # 1a,1c Establish the following differentiation formulas:

$$(\sin z)' = \cos z \quad \& \quad (\sinh z)' = \cosh z$$

Solution Using the complex definition of these functions:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sinh z = \frac{e^{z} - e^{-z}}{2}, \quad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

and chain rule with linearity of the derivative:

$$\frac{d}{dz} e^{f(z)} = e^{f(z)} \frac{df}{dz}, \quad \frac{d}{dz}(f(z) + g(z)) = \frac{df}{dz} + \frac{dg}{dz}$$

the claim follows immediately. $\square$

2.1 - # 2,4,6 Find the derivative of

$$z^2 + 10z, \quad [\cos(z^2)]^3, \quad (\log z)^3 \quad \text{on } \mathbb{C} \setminus \{x \leqslant 0\}$$

Solution 2)We've seen that

$$\frac{d}{dz} z^n = nz^{n-1}$$

Thus

$$\frac{d}{dz}(z^2 + 10z) = 2z + 10$$

4) Using chain rule and the previous question, we see

$$\frac{d}{dz}[\cos(z^2)]^3 = 3[\cos(z^2)]^2 * [\cos(z^2)]' = 3[\cos(z^2)]^2 * \sin(z^2) * 2z = 6z \sin(z^2)[\cos(z^2)]^2$$

6) We've seen that

$$\frac{d}{dz} \log z = \frac{1}{z}$$

Thus

$$\frac{d}{dz}(\log z)^3 = 3(\log z)^2 (\log z)' = 3\frac{(\log z)^2}{z}$$

2.1 - # 15 Let f be analytic on a domain D and suppose that $f'(z) = 0$ for all $z \in D$. Show that f is constant on D.

Solution Since f is analytic, we know that

$$f(z) = f(z_0) + \int_{z_0}^{z} \underbrace{f'(\zeta)}_{=0} d\zeta = f(z_0), \quad \forall z \in D$$

Thus f is constant on D □

2.1 - # 18 Show that $h(z) = \bar{z}$ is not analytic on any domain

Solution We check the Cauchy Riemann Equations: If $f(z) = u(x, y) + iv(x, y)$ and

$$\partial_x u = \partial_y v \quad \& \quad \partial_y u = -\partial_x v$$

with the partials continuous, then f is analytic. In our case we have $h(z) = \bar{z} = x - iy$, so

$$u(x, y) = x \quad \& \quad v(x, y) = -y$$

but

$$(\partial_x u =)1 \neq -1(= \partial_y v)$$

so h is not analytic on any domain. □

2.1 - # 20a,20e Let $f = u + iv$ be analytic. In each of the following, find v given u:

$$u = x^2 - y^2, \quad \& \quad u = \cosh x \cos y$$

Solution a)We need f to satisfy the Cauchy Riemann Equations. Thus we have

$$\partial_x u = 2x = \partial_y v \quad \& \quad \partial_y u = -2y = -\partial_x v \implies \begin{cases} 2y = \partial_x v \\ 2x = \partial_y v \end{cases}$$

Integrating the equations we've found, we see

$$v = 2xy + C \quad \mathbb{C}$$

Notice that

$$f = u + iv = z^2 + C$$

an equivalent version of the CR equations is that $\partial_{\bar{z}} f(z) = 0$. e)We need f to satisfy the Cauchy Riemann Equations. Thus we have

$$\partial_x u = \sinh x \cos y = \partial_y v \quad \& \quad \partial_y u = -\cosh x \sin y = -\partial_x v \implies \begin{cases} \cosh x \sin y = \partial_x v \\ \sinh x \cos y = \partial_y v \end{cases}$$

Integrating the equations we've found, we see

$$v = \sinh x \sin y + C \quad C \in \mathbb{C}$$

Notice that

$$f = u + iv = \cosh(z) + C$$

we see no $\bar{z}$ again, so it's analytic. □