Test 1

MAT334 – Complex Variables – Spring 2016

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SOLUTIONS

Question 1 Let D be the open disc centred at i and radius 3. Prove the following statements:

- $|z - w| < 6, \quad \forall z, w \in D$
- Show that the exponential function $z \to e^z$ is one-to-one on D

Solution Since $|z - i| < 3$ and $|w - i| < 3$ if $z, w \in D$, we see by the triangle inequality

$$|z - w| = |z - i - (-i + w)| \leqslant |z - i| + |w - i| < 3 + 3 = 6$$

Assume that $z = x_1 + iy_1$ and $w = x_2 + iy_2$, then we see

$$e^z = e^w \implies e^{x_1}e^{iy_1} = e^{x_2}e^{iy_2} \implies e^{x_1 - x_2}e^{i(y_1 - y_2)} = 1 \implies \begin{cases} x_1 - x_2 = 0 \\ y_1 - y_2 = 2\pi k, \quad k \in \mathbb{Z} \end{cases}$$

Thus we see $x_1 = x_2$ and $y_1 = y_2 + 2\pi k$, but since $z, w \in D$, we have

$$|y_1 - y_2| = |2\pi k| < 6 \implies k = 0 \implies y_1 = y_2 \implies z = w$$

$\square$

Question 2 Determine all the possible values of $(\sqrt{3} + i)^i$, specify which quadrant(s) of the plane contains these values.

Solution By definition of complex exponentiation, we have

$$(\sqrt{3} + i)^i = \exp(i \log(\sqrt{3} + i)) = \exp(i(\log(2) + i \arg(\sqrt{3} + i))) = \exp\left(i \log(2) - \frac{\pi}{6} + 2\pi k\right), \quad k \in \mathbb{Z}$$

Since $\log 2 \in (0, \pi/2)$. We know the the values lie in the 1st quadrant of the plane. $\square$

Question 3 Compute the sum of the series

$$\sum_{n=1}^{\infty} \frac{2 + i}{(1 + i)^n}$$

write the result in the form $a + ib$ where $a, b \in \mathbb{R}$

Solution Notice the series is geometric, so we use

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Thus

$$\sum_{n=1}^{\infty} \frac{2+i}{(1+i)^n} = \frac{2+i}{1-1/(1+i)} - (2+i) = \frac{(2+i)(1+i)}{i} - 2 - i = 3 - i - 2 - i = 1 - 2i$$

□

Question 4 Evaluate the line integral

$$\int_\gamma \sqrt{\bar{z}}dz$$

where γ is the upper-half of the unit circle, with the counterclockwise orientation. (Here we consider the branch for which $\sqrt{1} = 1$).

Solution We parametrize the path using

$$\gamma(\theta) = e^{i\theta} \quad \theta \in [0, \pi)$$

so $\gamma'(\theta) = ie^{i\theta}$, thus

$$\int_\gamma \sqrt{\bar{z}}dz = \int_0^\pi \sqrt{e^{-i\theta}} ie^{i\theta} d\theta = i \int_0^\pi \underbrace{e^{-i\theta/2+i\theta}}_{\sqrt{1}=1} d\theta = i \int_0^\pi e^{i\theta/2} d\theta = 2\left[e^{i\theta/2}\right]_0^\pi = 2(i-1)$$

□