Tutorial Problems #4

MAT 292 – Calculus III – Fall 2015

SOLUTIONS

Q. Consider the DE $y' = f(y)$ where the function $f(y)$ is differentiable. Assume also that $f(y_1) = f(y_2) = 0$ and $y_1 < y_2$.

(a) If the equilibrium solution $y = y_1$ is stable, than what do we know about $f(y)$ around the point y_1?

(b) Assume that both equilibria $y = y_1$ and $y = y_2$ are stable. Show that there must be another equilibria point y^* such that $y_1 < y^* < y_2$ and $y = y^*$ is unstable.

Solution

(a) We know if $y = y_1$ is stable, then we have for small positive ϵ that

$$f(y_1 - \epsilon) > 0 \quad \& \quad f(y_1 + \epsilon) < 0$$

(b) Since $y = y_1$ and $y = y_2$ are stable, we have for small positive ϵ that

$$f(y_1 + \epsilon) < 0 \quad \& \quad f(y_2 - \epsilon) > 0$$

We also know that $f(y)$ is continuous, thus the Intermediate value theorem gives us that there exists $y^* \in (y_1, y_2)$ such that

$$f(y^*) = 0 \quad \& \quad f(y^* - \epsilon') < 0 \quad \& \quad f(y^* + \epsilon') > 0$$

i.e. $y = y^*$ is an unstable equilibrium. □

2.4 - # 24 Consider the equation

$$dy/dt = ay - y^3 = y(a - y^2)$$

(a) Again consider the cases $a < 0, a = 0$ and $a > 0$. In each case, find the critical points, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable.

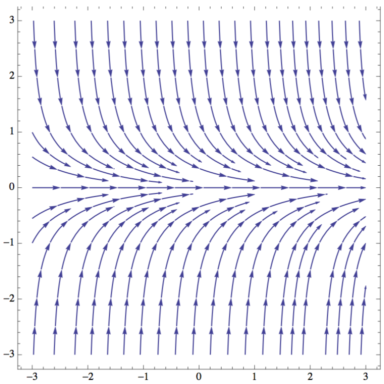
¶Recall that a critical point is simply $y' = 0$, thus

$$y' = 0 \iff y = 0 \quad \text{or} \quad a - y^2 = 0 \implies y = \pm\sqrt{a}$$

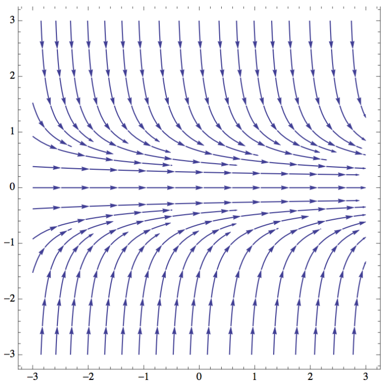
If $a < 0$ we have that $y = 0$ is the only critical point. If $a = 0$, we again have $y = 0$. If $a > 0$, we have the two roots $\pm\sqrt{a}$ and $y = 0$.

(b) In each case, sketch several solutions of the ODE in the y-plane

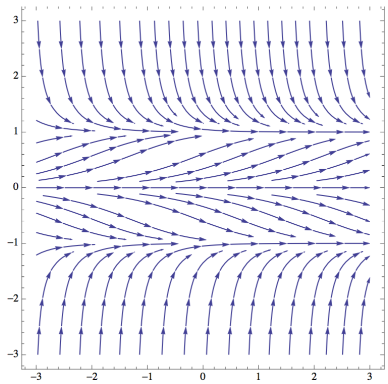
¶We sketch the resulting phase portraits.



$a < 0$

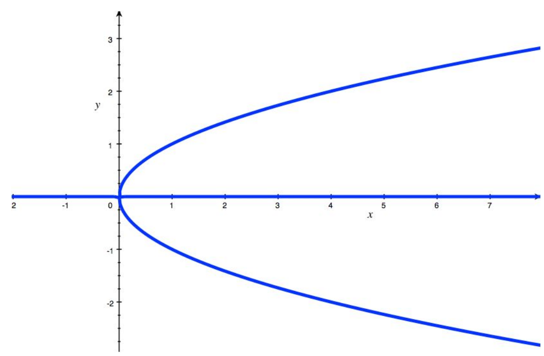


$a = 0$



$a > 0$

(c) Draw the bifurcation diagram for the ODE. Note that $a = 0$ is a pitch fork bifurcation.



2.5 - # 23 Show that if $(N_x - M_y)/M = Q$, where Q is a function of y only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form

$$\mu(y) = \exp \int Q(y)dy$$

Solution Suppose that $M + Ny' = 0$ is not exact and consider

$$\underbrace{\mu(y)M}_{\bar{M}}\,dx + \underbrace{\mu(y)N}_{\bar{N}}\,dy = 0$$

We'll try to find the condition on μ to make this exact. How do we do this? Check $M'_y = N'_x$.

$$\bar{M}_y = \frac{\partial}{\partial y}(\mu(y)M) = \mu'(y)M + \mu(y)M_y$$

$$\bar{N}_x = \frac{\partial}{\partial x}(\mu(y)N) = \mu(y)N_x$$

Using these equations, we can form an ODE in μ. Namely

$$0 = \bar{N}_x - \bar{M}_y = \mu(y)(N_x - M_y) - \mu'(y)M \iff \frac{\mu'(y)}{\mu(y)} = \frac{N_x - M_y}{M} = Q$$

By solving the above ODE for μ, we obtain

$$\boxed{\mu(y) = \exp \int Q(y)dy}$$

2.5 - # 26 Find an integrating factor and solve the given equation

$$y' = e^{2x} + y - 1$$

Solution Rewrite the ODE in differential form

$$\underbrace{(e^{2x} + y - 1)}_{M} dx + \underbrace{(-1)}_{N} dy = 0$$

We check the partials.

$$M_y = 1$$

$$N_x = 0$$

Since the equation is not exact, we'll need an integrating factor. Following the same logic as the previous question, we deduce

$$\mu(x) = \exp \int \left(\frac{M_y - N_x}{N} \right) dx = \exp \left(- \int dx \right) = e^{-x}$$

will work. Let's check

$$\underbrace{(e^{x} + e^{-x}(y - 1)}_{\bar{M}} dx + \underbrace{(-e^{-x})}_{\bar{N}} dy = 0$$

$$\bar{M}_y = e^{-x}$$

$$\bar{N}_x = e^{-x}$$

Now the equation is exact! Thus we can just integrate each part respectively.

$$\int \bar{M} dx = \int (e^{x} + e^{-x}(y - 1))dx = e^{x} + e^{-x}(1 - y) + C(y)$$

$$\int \bar{N} dy = \int -e^{-x} dy = -ye^{-x} + \tilde{C}(x)$$

By comparing the above equation, we see that a function satisfying the partials is

$$F(x, y) = e^{x} + e^{-x}(1 - y)$$

This implies the general solution is

$$\boxed{const = e^{x} + e^{-x}(1 - y)}$$

2.4 - # 18 A pond forms as water collects in a conical depression of radius a and depth h. Suppose that water flows in at a constant rate k and is lost through evaporation at a rate proportional to the surface area.

(a) Show that the volume $V(t)$ of water in the pond at time t satisfies the differential equation

$$dV/dt = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3}$$

where α is the coefficient of evaporation

¶The model we'd like to use is

$$\frac{dV}{dt} = V_{in} - V_{out}$$

we're given that $V_{in} = k$, and that $V_{out} = \alpha SA$ (out of the top, i.e. just a circle). We just have to compute the surface area of the cone in terms of it's Volume. Recall that

$$V_{cone} = \frac{\pi r^2 l}{3} \quad \& \quad SA_{circle} = \pi r^2$$

where r is radius and l is the length. By drawing a picture, you'll find that the ratio between the length and radius is always the same i.e. $l/r = h/a$. Thus we have

$$V_{cone} = \frac{\pi r^2 l}{3} = \frac{\pi r^3 h}{3a} \implies \sqrt[3]{\frac{3aV_{cone}}{\pi h}} = r$$

$$\implies SA = \pi\left(\frac{3aV_{cone}}{\pi h}\right)^{2/3}$$

Therefore, the ODE is

$$\boxed{dV/dt = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3}}$$

(b) Find the equilibrium depth of water in the pond. Is the equilibrium asymptotically stable?

¶Recall that equilibrium occurs when $V' = 0$, so we have to find the roots of the ODE. We see

$$\frac{dV}{dt} = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3} = 0 \iff \boxed{V = \pm\frac{(k/\alpha\pi)^{3/2}\pi h}{3a}}$$

Since the Volume cannot be negative, we discard that root. To find the depth l, just substitute back in as in the previous part.

(c) Find a condition that must be satiated if the pond is not to overflow.

¶For the pond to not overflow, we need $dV/dt = 0$ when the cone is full. Thus

$$V_{cone} = \frac{\pi a^2 h}{3} = \frac{(k/\alpha\pi)^{3/2}\pi h}{3a} \implies \boxed{k = \alpha\pi a^{4/3}}$$