Tutorial Problems #9

MAT 292 – Calculus III – Fall 2014

SOLUTIONS

4.5.11 We look separately at the equations:

$$y'' + y' + 4y = e^t, \tag{1}$$

and

$$y'' + y' + 4y = e^{-t}, \tag{2}$$

since $2\sinh(t) = e^t - e^{-t}$.
For (1) we look for a special solution of the form Ae^t. Substituting this into (1) we get that $A = 1/6$.
For (2) we look for a special solution of the form Be^{-t}. Substituting this into (2) we get that $B = -1/4$.
Since the general solution of the linear equation

$$y'' + y' + 4y = 0$$

is given by

$$y_l(t) = c_1 e^{-t/2}\cos(\sqrt{15}/2) + c_2 e^{-t/2}\sin(\sqrt{15}/2),$$

as $\frac{-1\pm i\sqrt{15}}{2}$ are the roots of $\lambda^2 + \lambda + 4 = 0$, we have that the general solution of the equation

$$y'' + y' + 4y = 2\sinh(t) \tag{3}$$

is given by

$$y(t) = c_1 e^{-t/2}\cos(\sqrt{15}/2) + c_2 e^{-t/2}\sin(\sqrt{15}/2) + \frac{1}{6}e^t - \frac{1}{4}e^{-t}.$$

4.5.27 a) Follows directly from substitution.
b) We use the method of integrating factors and we have that:

$$w(t) = e^{5t}\int 2e^{-5t}dt + Ce^{5t} = Ce^{5t} - \frac{2}{5}. \tag{4}$$

c) Integrating (4) we get that:

$$v(t) = \frac{1}{5}Ce^{5t} - \frac{2}{5}t + C_0.$$

Then we have as required that:

$$Y(t) = v(t)e^{-t} = -\frac{2}{5}te^{-t} + \frac{1}{5}Ce^{4t} + C_0e^{-t}.$$

4.5.30 The change of variables $t = \ln x$ gives us that:

$$x\frac{dy}{dx} = \frac{dy}{dt}, \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$$

By denoting now by y' the t derivative of y (i.e. $y' = \frac{dy}{dt}$) we have that our equation turns into the following one:

$$y'' - 3y' + 2y = 3e^{2t} + 2t. \tag{5}$$

The solution for the linear part of (5) is given by:

$$y_l(t) = c_1 e^t + c_2 e^{2t}.$$

For a general solution of (5) we consider separately the equations:

$$y'' - 3y' + 2y = 3e^{2t}, \tag{6}$$

and

$$y'' - 3y' + 2y = 2t. \tag{7}$$

For (6) we look for a special solution of the form Ate^{2t}. Substituting this into (6) we get that $A = 3$.
For (7) we look for a special solution of the form $B_1 t + B_2$. Substituting this into (7) we get that $B_1 = 1$, $B_2 = 3/2$.
By converting back to the x variable, we find that a general solution of (5) is given by:

$$y(x) = c_1 x + c_2 x^2 + 3x^2 \ln x + \ln x + \frac{3}{2}.$$

4.7.28 We use as before the change of variables $x = \ln t$ (which is permissible by the range of t). Then with $y' = \frac{dy}{dx}$ we have the equation:

$$y'' - y' - 2y = 3e^{2x} - 1. \tag{8}$$

The solution of its linear part is given by:

$$y_l(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 e^{-x} + c_2 e^{2x}.$$

From this we can compute the Wronskian:

$$W[y_1, y_2](x) = 3e^x.$$

Now we seek a special solution of (8) of the form:

$$Y(x) = u_1(x) y_1(x) + u_2(x) y_2(x).$$

Using the equation (or directly Theorem 4.7.2, or formulas (25) in page 289 of the textbook) we see that we have:

$$u_1(x) = -\frac{e^{2x}(3e^{2x} - 1)}{3e^x} = -\frac{1}{3}e^{3x} + \frac{1}{3}e^x,$$
$$u_2(x) = \frac{e^{-x}(3e^{2x} - 1)}{3e^x} = x + \frac{1}{6}e^{-2x}.$$

Integrating these two equations we get that $Y(t)$ has the form:

$$Y(x) = xe^{2x} - \frac{1}{3}e^{2x} + \frac{1}{2}.$$

By switching back to the t variable we get that the general solution of (8) has the form:

$$y(t) = C_1 \frac{1}{t} + C_2 t^2 + t^2 \ln t + \frac{1}{2}.$$

4.7.39 The equation for v follows by a direct substitution. We let $w = v'$. We have that w satisfies the following equation:

$$w' + P(t)w = Q(t), \tag{9}$$

where

$$P(t) = \frac{2y_1'(t) + p(t)y_1(t)}{y_1(t)} \text{ and } Q(t) = \frac{g(t)}{y_1(t)}.$$

We use the method of integrating factors for this 1st order equation and we have that:

$$w(t) = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)Q(s)ds + \frac{C}{\mu(t)},$$

where

$$\mu(t) = exp\left(-\int_{t_0}^{t} P(s)ds\right).$$

Letting

$$F(t) = \int_{t_0}^{t} \mu(s)Q(s)ds,$$

we have that:

$$v(t) = \int_{t_0}^{t} w(s)ds + v(t_0) = \int_{t_0}^{t} \frac{F(s) + C}{\mu(s)} ds + v(t_0),$$

which then gives us a general formula for the required $y(t) = v(t)y_1(t)$.