Tutorial Problems #7

MAT 292 – Calculus III – Fall 2014

SOLUTIONS

3.5.14.

(a) With $L = 4R^2C$ we note that the determinant of

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 0-\lambda & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC}-\lambda \end{pmatrix},$$

is given by

$$\lambda^2 + \frac{\lambda}{RC} + \frac{1}{LC} = \left(\lambda + \frac{1}{2RC}\right)^2,$$

so that

$$\left(\lambda + \frac{1}{2RC}\right)^2 = 0 \Rightarrow \lambda = -\frac{1}{2RC}.$$

(b) With $R = 1$, $C = 1$, $L = 4$ we have that:

$$\lambda = -\frac{1}{2}.$$

A corresponding eigenvector is

$$\mathbf{v} = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}.$$

We solve the system:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v},$$

and a corresponding generalized eigenvector is

$$\mathbf{w} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

We arrive at the general solution

$$\mathbf{x}(t) = c_1 e^{-t/2}\mathbf{v} + c_2(te^{-t/2}\mathbf{v} + e^{-t/2}\mathbf{w}).$$

By the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

we get that $c_1 = c_2 = -2$.

3.5.16. We find the eigenvalues:

$$\begin{vmatrix} a_{11}-r & a_{12} \\ a_{21} & a_{22}-r \end{vmatrix} = 0 \quad \Leftrightarrow \quad (a_{11}-r)(a_{22}-r) - a_{12}a_{21} = 0$$

$$\Leftrightarrow \quad r^2 - pr + q = 0 \quad \Leftrightarrow \quad r = \frac{p \pm \sqrt{\Delta}}{2}.$$

(a) If $q > 0$ and $p < 0$, then $\Delta = p^2 - 4q < p^2$ and there are two options:

- If $\Delta > 0$, then both eigenvalues are real and negative:

$$r_1 = \frac{p - \sqrt{\Delta}}{2} < \frac{p}{2} < 0 \qquad \text{and} \qquad r_2 = \frac{p + \sqrt{\Delta}}{2} < \frac{p + \sqrt{p^2}}{2} = 0$$

- If $\Delta < 0$, then the eigenvalues are complex with real part $\frac{p}{2} < 0$.

In either case, the solutions are asymptotically stable.

(b) If $q > 0$ and $p = 0$, then $\Delta = -4q < 0$ and the eigenvalues are complex with no real part, so the critical point $(0, 0)$ is a center, which is stable.

(c) We have two options

- If $q < 0$, then $\Delta = p^2 - 4q > p^2 > 0$ and the eigenvalues are real and have opposite signs:

$$r_1 = \frac{p - \sqrt{\Delta}}{2} < \frac{p - \sqrt{p^2}}{2} \leqslant 0 \qquad \text{and} \qquad r_2 = \frac{p + \sqrt{\Delta}}{2} > \frac{p + \sqrt{p^2}}{2} \geqslant 0$$

  So $(0, 0)$ is a saddle-node, which is unstable.

- If $p > 0$, then there are 5 cases:
  - If $\Delta < 0$, then the eigenvalues are complex with real part $\frac{p}{2} > 0$. Then $(0, 0)$ is a spiral source, which is unstable.
  - If $\Delta = 0$, then there is only 1 eigenvalue: $\frac{p}{2} > 0$, so $(0, 0)$ is an unstable improper node.
  - If $0 \leqslant \Delta < p^2$, then the eigenvalues are real and positive. Then $(0, 0)$ is an unstable node.
  - If $\Delta = p^2$, then the eigenvalues are 0 and $p > 0$. So $(0, 0)$ is unstable.
  - If $\Delta > p^2$, then the eigenvalues are real and have opposite signs. So $(0, 0)$ is a saddle-node, which is unstable.

6.2.10.(a) W is given by

$$W(t) = Cexp\left(\int_{t_0}^{t} \operatorname{tr}\big(\mathbf{P}(s)\big)ds\right).$$

If W is zero or not depends on the initial condition thus agreeing with Theorem 6.2.5 and Theorem 6.2.1 which asserts uniqueness of the solution when $\mathbf{P}(t)$ is continuous (as it is assumed here).

6.5.6. By 3.4.7 (from tutorial #6), we have that a fundamental matrix is given by:

$$\mathbf{X}(t) = \begin{pmatrix} -2e^{-t}\sin(2t) & 2e^{-t}\cos(2t) \\ e^{-t}\cos(2t) & e^{-t}\sin(2t) \end{pmatrix}.$$

Computing $\mathbf{X}(0)$ and then the inverse of that, we get that:

$$e^{\mathbf{A}t} = \mathbf{\Phi}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(0) = \begin{pmatrix} e^{-t}\cos(2t) & -2e^{-t}\sin(2t) \\ \frac{1}{2}e^{-t}\sin(2t) & e^{-t}\cos(2t) \end{pmatrix},$$

as

$$\mathbf{X}^{-1}(0) = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

6.5.15. We just computed the special fundamental matrix, so the solution required is

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\cos(2t) - 2\sin(2t) \\ \frac{3}{2}\sin(2t) + \cos(2t) \end{pmatrix} e^{-t}.$$

6.5.15. (extra) Just like before, we have

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\cos(2t) - 4\sin(2t) \\ \sin(2t) + 2\cos(2t) \end{pmatrix} e^{-t}.$$

Remark. These last 3 exercises are meant to show the advantage of computing the special fundamental matrix $\mathbf{\Phi}$ when we need to apply different initial conditions to the same system of differential equations: once $\mathbf{\Phi}$ is computed, it is a simple matter to find solutions to different initial conditions.