Tutorial Problems #13

MAT 292 – Calculus III – Fall 2014

SOLUTIONS

9.2.13. The Fourier series are given by:

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x/L),$$

where

$$\begin{aligned} b_n &= -\frac{1}{L}\int_{-L}^{L} x\sin(n\pi x/L)\,dx \\ &= \left[x\frac{\cos(n\pi x/L)}{n\pi}\right]_{-L}^{L} - \frac{1}{n\pi}\int_{-L}^{L}\cos(n\pi x/L)\,dx \\ &= \frac{2L}{n\pi}\cos(n\pi) - \frac{L}{(n\pi)^2}\left[\sin(n\pi x/L)\right]_{-L}^{L} \\ &= \frac{2L(-1)^n}{n\pi} \end{aligned}$$

9.2.extra. The function:

$$f(x) = \cos(3x) - \cos(8x)$$

has period 2π. So we can take its Fourier expansion in $[-\pi, \pi]$ that will be of the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx).$$

But the function is already in this form: the only non-zero coefficients will be $a_3 = 1$ and $a_8 = -1$.

9.6.13.

(a) Follow the procedure in pages 664–666 to obtain the solution

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{40}\right) e^{-\left(\frac{n\pi}{80}\right)^2 t},$$

where

$$\begin{aligned} c_0 &= \frac{2}{40}\int_0^{40} \frac{x(60-x)}{30}\,dx = \frac{400}{9} \\ c_n &= \frac{2}{40}\int_0^{40} \frac{x(60-x)}{30}\cos\left(\frac{n\pi x}{40}\right)\,dx \end{aligned}$$

(c) As we let $t \to \infty$, we obtain the steady-state temperature, which is

$$\lim_{t\to\infty} u(x,t) = \frac{c_0}{2} = \frac{200}{9},$$

which is the average of the initial temperature of the bar.

(d) The solution has the form:

$$u(t,x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t/6400}\cos(n\pi x/40).$$

so we need to find t such that

$$\left| u(x,t) - \frac{c_0}{2} \right| < 1.$$

This is equivalent to

$$\underbrace{\left| \sum_{n=1}^{\infty} \left[c_n e^{-n^2\pi^2 t/6400}\cos(n\pi x/40) \right] \right|}_{=S} \leqslant 1. \tag{$\star$}$$

The goal is to estimate the series on the left-hand side and try to obtain a geometric series that is larger than the original S.

We have

$$\begin{aligned} S &\leqslant \sum_{n=1}^{\infty} \left| c_n e^{-n^2\pi^2 t/6400}\cos(n\pi x/40) \right| \\ &\leqslant \sum_{n=1}^{\infty} |c_n| e^{-n^2\pi^2 t/6400} \\ &\leqslant \sum_{n=1}^{\infty} |c_n| e^{-n\pi^2 t/6400} \end{aligned}$$

where on the last step we used the fact that $n \leqslant n^2$.

Then, we need to estimate $|c_n|$:

$$\begin{aligned} |c_n| &= \frac{2}{40}\left| \int_0^{40} \frac{x(60-x)}{30} \cos\left(\frac{n\pi x}{40}\right) dx \right| \\ &\leqslant \frac{2}{40}\int_0^{40} \left| \frac{x(60-x)}{30} \right| dx \\ &\leqslant \frac{1}{20} \max_{x\in[0,40]} \frac{x(60-x)}{30}(40-0) \\ &= \frac{120}{20} = 6. \end{aligned}$$

Then we get

$$S \quad \leqslant \quad 6\sum_{n=1}^{\infty} e^{-n\pi^2 t/6400} \quad = \quad 6e^{-\pi^2 t/6400}\frac{1}{1-e^{-\pi^2 t/6400}} \quad = \quad \frac{6}{e^{\pi^2 t/6400}-1}.$$

We now want to check when this quantity is smaller than 1:

$$\begin{aligned}\frac{6}{e^{\pi^2 t/6400}-1} &\leqslant 1\\ e^{\pi^2 t/6400} &\geqslant 7\\ \frac{\pi^2 t}{6400} &\geqslant \ln 7\\ t &\geqslant \frac{6400\ln 7}{\pi^2} \approx 1262\, s.\end{aligned}$$

Note. This is not the best t. This is a value for t, after which we can prove that the temperature is within 1^o of the steady-state temperature.
Many of the estimates here are not very good, they are just good enough to give us an estimate.

9.6.11a. To find the solution, we need to write

$$u(x,t) = u_E(x) + v(x,t),$$

where $u_E(x)$ is the equilibrium solution and $v(x,t)$ is the offset from equilibrium.
Then, since $u_E(x)$ depends on x only, it satisfies $u_E''(x) = 0$, which gives $u_E(x) = ax + b$. Using the initial conditions, we deduce that

$$u_E(x) = 30 - x.$$

Then the offset $v(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2} \\ v(0,t) = 0 \\ v(30,t) = 0 \\ v(x,0) = x(60-x)/30 - (30-x) = \overline{f}(x). \end{cases}$$

This is the solution of the Heat Equation with "frozen ends", which we computed in lecture:

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{30}\right) e^{(\frac{n\pi}{L})^2 t},$$

where

$$b_n = \frac{2}{30}\int_0^{30} \overline{f}(x)\sin(n\pi x/30)\, dx.$$