Tutorial Problems #7

MAT 267 – Advanced Ordinary Differential Equations – Fall 2016

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Solutions

Solving a Linear System with Constant Coefficients Suppose you want to solve

$$\dot{x} = Ax \quad w/ \quad x(t_0) = x_0$$

Recall that If A is non-defective (algebraic multiplicity = geometric multiplicity), then there exists $\Lambda \in GL(n, \mathbb{R})$ (i.e. the eigenvectors) and a diagonal matrix D (of the eigenvalues of A) s.t.

$$A = \Lambda D \Lambda^{-1}$$

Now since Λ is full of constants, we see that

$$\dot{x} = Ax \iff \frac{d}{dt}(\Lambda^{-1}x) = D\Lambda^{-1}x$$

Thus if we let $y = \Lambda^{-1}x$, the system decouples :

$$\dot{x} = Ax \iff \dot{y} = Dy \iff y_i' = \lambda_i y \implies y_i(t) = C_i e^{\lambda_i t}$$

Now changing back into x, we obtain that

$$x = \Lambda y = C_1 \vec{\lambda}_1 e^{\lambda_1 t} + \ldots + C_n \vec{\lambda}_n e^{\lambda_n t}$$

Thus to solve a non-defective system, we simply need the eigenvalues and eigenvectors of A. If A is defective, we know there exists a Jordan matrix J and a $\Lambda \in GL(n, \mathbb{R})$ s.t.

$$A = \Lambda J \Lambda^{-1}$$

Thus the same procedure applies except the system doesn't fully decouple i.e. on the Jordan blocks we have

$$y_n' = \lambda y_n \quad \& \quad y_i' = \lambda y_i + y_{i+1}$$

We may solve these inductively to obtain that

$$y^{(1)} = \begin{pmatrix} e^{\lambda t} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, y^{(2)} = \begin{pmatrix} te^{\lambda t} \\ e^{\lambda t} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, y^{(n)} = \begin{pmatrix} t^{n-1}e^{\lambda t}/(n-1)! \\ t^{n-2}e^{\lambda t}/(n-2)! \\ t^{n-3}e^{\lambda t}/(n-3)! \\ \vdots \\ e^{\lambda t} \end{pmatrix},$$

Thus the solution to the system is simply

$$x = \Lambda y$$

Generalized Eigenvectors. Without going into too much detail, the missing eigenvectors are replaced with generalized eigenvectors (you have as many as you're missing from eigenvector deficiency). You want a generalized eigenvector of rank k to satisfy

$$(A-\lambda 1)^k \lambda_g^k = 0 \quad \text{but} \quad (A-\lambda 1)^{k-1}\lambda_g^k \neq 0$$

The easiest way to satisfy the above is to just take linear combinations of eigenvectors and generalized eigenvectors (as you move up rank). i.e. take $(A-\lambda 1)\lambda_g^k = \lambda_g^{k-1}$ with the convention that $\lambda_g^0 = \vec{\lambda}$.

Example Solve

$$\dot{x} = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 2 \end{pmatrix} x \quad w/ \quad x_0 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Solution As the above, let's find the eigenvalues of the matrix. We see

$$P(\lambda) = \det(A - 1\lambda) = (2-\lambda)^2$$

Thus we see $\lambda = 2$ with algebraic multiplicity of 3. Now what about the eigenvectors? We see that

$$\ker(A-1\lambda) = \ker \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 3 & 2 & 0 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \implies \vec{\lambda} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus we're missing two eigenvectors (i.e. the geometric multiplicity is only 1)! So we compute generalized eigenvectors. We see

$$(A-1\lambda)\vec{\lambda}_{g_1} = \vec{\lambda} \iff \vec{\lambda}_{g_1} \in \left\{ \begin{pmatrix} 1 \\ -1 \\ s \end{pmatrix} : s \in \mathbb{R} \right\} \implies \vec{\lambda}_{g_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Next up we see

$$(A-1\lambda)\vec{\lambda}_{g_2} = \vec{\lambda}_{g_1} \iff \vec{\lambda}_{g_2} \in \left\{ \begin{pmatrix} -2 \\ 3 \\ s \end{pmatrix} : s \in \mathbb{R} \right\} \implies \vec{\lambda}_{g_2} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$$

Thus our matrix Λ takes the form

$$\Lambda = \begin{pmatrix} 0 & 1 & -2 \\ 0 & -1 & 3 \\ 1 & 0 & 0 \end{pmatrix} \implies A = \Lambda \underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}}_{=J} \Lambda^{-1}$$

From our previous computation, we know the general solution is

$$x(t) = \Lambda(c_1 y^{(1)} + c_2 y^{(2)} + c_3 y^{(3)}) = \Lambda \left(c_1 \begin{pmatrix} e^{2t} \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} te^{2t} \\ e^{2t} \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} t^2 e^{2t}/2 \\ te^{2t} \\ e^{2t} \end{pmatrix} \right)$$

The initial data implies the constants must satisfy

$$\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0 & 1 & -2\\0 & -1 & 3\\1 & 0 & 0\end{pmatrix}\left(c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix}\right) \implies c_1 = 1 \quad c_2 = 2 \quad c_3 = 1$$

Thus the solution to the IVP is

$$\boxed{x(t) = e^{2t}\begin{pmatrix}t\\1-t\\1+2t+t^2/2\end{pmatrix}}$$

Matrix Exponentials Define via formal power series

$$\exp(At) = \sum_{n=0}^{\infty}\frac{(At)^n}{n!}$$

Notice that this solves $\dot{x} = Ax$ if A is constant

$$\dot{x} = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{(At)^n}{n!} = \sum_{n=1}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{A^{n+1}t^n}{n!} = A\sum_{n=0}^{\infty}\frac{(At)^n}{n!} = Ax$$

notice that $\exp(A * 0) = 1$. Define $\Phi(t) = X(t)X^{-1}(0)$ where X solves $\dot{X} = AX$ aka the fundamental matrix solution, notice that $\phi(0) = 1$. By uniqueness of solutions to this problem (See Picard-Lindelöf Theorem), we conclude

$$\exp(At) = X(t)X^{-1}(0)$$

or more generally,

$$\exp(A(t-t_0)) = X(t)X^{-1}(t_0)$$

Matrix Exponential Question Find e^{At} for

$$A = \begin{pmatrix}-1 & -4\\1 & 1\end{pmatrix}$$

Solution Using the techniques we've learned to date, one may show that

$$X(t) = e^{-t}\begin{pmatrix}-2\sin(2t) & 2\cos(2t)\\\cos(2t) & \sin(2t)\end{pmatrix} \quad \text{solves} \quad \dot{X} = AX$$

Using the above formula, we compute

$$X(0) = \begin{pmatrix}0 & 2\\1 & 0\end{pmatrix} \implies X^{-1}(0) = \begin{pmatrix}0 & 1\\1/2 & 0\end{pmatrix}$$

Thus the exponential is given by

$$e^{At} = X(t)X^{-1}(0) = e^{-t}\begin{pmatrix}-2\sin(2t) & 2\cos(2t)\\\cos(2t) & \sin(2t)\end{pmatrix}\begin{pmatrix}0 & 1\\1/2 & 0\end{pmatrix} = e^{-t}\begin{pmatrix}\cos(2t) & -2\sin(2t)\\\sin(2t)/2 & \cos(2t)\end{pmatrix}$$

□

Quiz Question Find e^{At} for

$$A = \begin{pmatrix} 5 & 1 \\ -1 & 3 \end{pmatrix}$$

Solution First we solve $\dot{x} = Ax$. Find the eigenvalues

$$P(\lambda) = \begin{vmatrix} 5-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = (\lambda - 4)^2 \implies \lambda = 4$$

We look for eigenvectors now, i.e. check out the kernel of $(A - 1\lambda)$

$$\ker \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \implies \vec{\lambda} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We're missing an eigenvector, so we find a generalized one.

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \vec{\lambda}_g = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \iff \vec{\lambda}_g \in \left\{ \begin{pmatrix} 1-s \\ s \end{pmatrix} : s \in \mathbb{R} \right\} \implies \vec{\lambda}_g = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus we have

$$\Lambda = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \& \quad J = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$$

We now pull back the solution to find that the fundamental solution is

$$X(t) = e^{4t}\Lambda \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = e^{4t} \begin{pmatrix} 1 & t \\ -1 & 1-t \end{pmatrix}$$

Note that

$$X(0) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \implies X^{-1}(0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Thus

$$e^{At} = X(t)X^{-1}(0) = e^{4t} \begin{pmatrix} 1 & t \\ -1 & 1-t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = e^{4t} \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}$$

Alternate Solution Notice that

$$A^n = 4^{n-1} \begin{pmatrix} 4+n & n \\ -n & 4-n \end{pmatrix}$$

Then by definition of the matrix exponential, we see that

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = \sum_{n=0}^{\infty} \begin{pmatrix} 4^{n-1}(4+n)/n! & n4^{n-1}/n! \\ -n4^{n-1}/n! & 4^{n-1}(4-n)/n! \end{pmatrix} t^n$$

Recall the taylor expansion for the exponential.

$$e^{at} = \sum_{n=0}^{\infty} \frac{a^n t^n}{n!}$$

Then clearly

$$\sum_{n=0}^{\infty} \frac{4^n t^n}{n!} = e^{4t} \quad \& \quad \sum_{n=0}^{\infty} \frac{n4^{n-1}t^n}{n!} = \sum_{n=1}^{\infty} \frac{4^{n-1}t^n}{(n-1)!} = t\sum_{n=1}^{\infty} \frac{4^{n-1}t^{n-1}}{(n-1)!} = t\sum_{n=0}^{\infty} \frac{4^n t^n}{n!} = te^{4t}$$

So the definition simplifies to

$$e^{At} = e^{4t} \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}$$

$\square$