Tutorial Problems #6

MAT 267 – Advanced Ordinary Differential Equations – Winter 2016

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Solutions

Variation of Parameters Suppose you know y_1 and y_2 solve $y'' + py' + qy = 0$. Is there a way to easily solve the non-homogeneous equation?

$$y'' + py' + qy = g$$

Yes!!! It turns out that if we try $y = A(t)y_1 + B(t)y_2$ (i.e. vary the parameters) it is a solution if

$$A(t) = -\int \frac{y_2 g}{W[y_1, y_2]} dt \quad \& \quad B(t) = \int \frac{y_1 g}{W[y_1, y_2]} dt$$

This is easily deduced from a straightforward computation assuming $A'y_1 + B'y_2 = 0$.

pg. 240 - # 5 Solve

$$y'' - 3y' + 2y = \cos\left(e^{-x}\right)$$

Solution First solve the homogenous part. i.e. notice that

$$L(D) = (D-2)(D-1)$$

Thus $\lambda = 1, 2$ are the eigenvalues and we have that

$$y_1(x) = e^{2x} \quad \& \quad y_2(x) = e^x$$

are the fundamental solutions. To now solve the non-homogeneous equation, we may use variation of parameters but we first need the Wronskian

$$W[y_1, y_2](x) = y_1 y_2' - y_1' y_2 = -e^{3x}$$

Using the formula we see that

$$\begin{aligned} A(x) &= \int \frac{e^x \cos e^{-x}}{e^{3x}} dx \\ &= -\int u \cos u du \quad \text{where} \quad u = e^{-x} \\ &= -u \sin u - \cos u + C_1 \\ &= -e^{-x} \sin e^{-x} - \cos e^{-x} + C_1 \end{aligned}$$

$$\begin{aligned} B(x) &= \int \frac{e^{2x}\cos(e^{-x})}{-e^{3x}}dx \\ &= \int \cos u dx \quad \text{where} \quad u = e^{-x} \\ &= \sin u + C_2 \\ &= \sin e^{-x} + C_2 \end{aligned}$$

Thus, we have the general solution as

$$\boxed{y(x) = A(x)y_1 + B(x)y_2 = C_1e^{2x} + C_2e^x - e^{2x}\cos e^{-x}}$$

□

Variation of Parameters in Higher Order Equations In general, if we have a first order system $\dot{x} = Ax + g$. You'll find that the fundamental solution X to $\dot{X} = AX$ allows us to write the solution as

$$x(t) = X(t)c + X(t)\int_{t_0}^{t} X^{-1}(s)g(s)d(s)$$

Indeed since

$$\dot{x} = \underbrace{\dot{X}c + \dot{X}\int_{t_0}^{t} X^{-1}(s)g(s)d(s)}_{\dot{X}=AX} + X(X^{-1}g) = A\left(Xc + X\int_{t_0}^{t} X^{-1}(s)g(s)d(s\right) + g = Ax + g$$

Notice we easily recover the formula we've been using in the 2nd order case since $\det X = W[y_1, y_2]$ and

$$g = \begin{pmatrix} 0 \\ g \end{pmatrix} \quad \& \quad X = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \implies X^{-1}g = \frac{1}{W[y_1, y_2]}\begin{pmatrix} y_2' & -y_2 \\ y_1' & y_1 \end{pmatrix}\begin{pmatrix} 0 \\ g \end{pmatrix} = \frac{1}{W}\begin{pmatrix} -y_2 g \\ y_1 g \end{pmatrix}$$

Reduction of Order when a solution is known If you know y_1 solves $y'' + py' + qy = 0$, then you may find y_2 by setting $y_2 = \nu(x)y_1(x)$ with a straight forward computation for $\nu(x)$. A nice way to go about find ν is though the Wronskian, since

$$W[y_1, y_2] = C\exp\left(-\int p(x)dx\right)$$

by Abel's theorem, and then by definition we have

$$W[y_1, y_2] = y_1y_2' - y_1'y_2 \iff \frac{W[y_1, y_2]}{y_1^2} = \frac{y_2'}{y_1} - \frac{y_2y_1'}{y_1^2} = \frac{d}{dx}\left(\frac{y_2}{y_1}\right)$$

Thus we see

$$y_2 = y_1\int \frac{W[y_1, y_2]}{y_1^2}dx$$

pg.246 - #16 Solve

$$x^2y'' - 2y = 2x^2 \quad \text{given} \quad y_1 = x^2$$

Solution In standard form the ODE is

$$y'' - \frac{2}{x^2}y = 2$$

Using the above, we know

$$y_2 = y_1 \int \frac{W}{y_1^2} dx$$

So we compute Wronskian via Abel's theorem

$$W[y_1, y_2] = c_1 \exp\left(-\int p(x)dx\right) = c_1$$

Now using the reduction of order formula we see

$$y_2(x) = x^2 \int \frac{dx}{x^4} = \frac{1}{x}$$

So the second fundamental solution to the ODE is $y_2 = 1/x$. Now that we have both solutions, let's use variation of parameters to solve the non-homogeneous part. i.e. $y(x) = A(x)y_1 + B(x)y_2$. We need to compute the the explicit Wronskian for our given fundamental solutions. We see

$$W[y_1, y_2](x) = y_1 y_2' - y_1' y_2 = -3$$

Now we use the variation of parameters formula

$$\begin{aligned} A(x) &= -\int \frac{y_2 g}{W} dx \\ &= \frac{2}{3}\int \frac{dx}{x} \\ &= \frac{2}{3}\log x + c_1 \\ B(x) &= \int \frac{y_1 g}{W} dx \\ &= -\frac{2}{3}\int x^2 dx \\ &= -\frac{2}{9}x^3 + c_2 \end{aligned}$$

Putting everything together now shows

$$\boxed{y(x) = c_1 x^2 + \frac{c_2}{x} + \frac{2}{3}x^2 \log(x)}$$

pg. 329 - # 5 Prove conservation of energy for the undamped helical spring ($mx'' = -kx$). i.e.

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \quad \text{where} \quad v = \frac{dx}{dt}$$

Solution Suppose that $x' \neq 0$, then we have

$$mx'' = -kx \implies mx''x' = -kxx' \implies \frac{1}{2}m\frac{d}{dt}(x')^2 = -\frac{1}{2}k\frac{d}{dt}x^2 \implies \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E \in \mathbb{R}$$

□

The Method of Undetermined Coefficients (Guessing the Answer) Suppose we have constant coefficients in some linear differential operator $L(D)$. Let $P(x)$ be an arbitrary polynomial of degree k, then the following non-homogeneous problems

$$L(D)y = \begin{cases} P(x) \\ P(x)\exp(ax) \\ P_1(x)\cos(bx) + P_2(x)\sin(bx) \\ P_1(x)e^{ax}\cos(bx) + P_2(x)e^{ax}\sin(bx) \end{cases}$$

have a particular solution in the form

$$y_p = \begin{cases} \tilde{P}(x) \\ \tilde{P}(x)\exp(ax) \\ \tilde{P}_1(x)\cos(bx) + \tilde{P}_2(x)\sin(bx) \\ \tilde{P}_1(x)e^{ax}\cos(bx) + \tilde{P}_2(x)e^{ax}\sin(bx) \end{cases}$$

where $\tilde{P}$ is a polynomial degree of $\tilde{k} = k$ if the fundamental solutions are linearly independent from $g(x)$, or $\tilde{k} > k$ if the fundamental solutions are dependent (add factors of x till you're not dependent). The proof follows from linearity and linear algebra exploiting the independence of the functions.

pg. 343 - #5 Solve

$$\frac{d^2y}{dt^2} + \omega_0^2 y = F\sin(\omega_0 t) \quad y(0) = y_0, v(0) = v_0$$

Solution Clearly the homogeneous part is

$$y_{hom}(t) = c_1 \underbrace{\cos(\omega_0 t)}_{=y_1} + c_2 \underbrace{\sin(\omega_0 t)}_{=y_2}$$

Since the RHS is simple, we know a particular solution takes the form

$$y_p = t[A\cos(\omega_0 t) + B\sin(\omega_0 t)] = t[Ay_1 + By_2]$$

Thus we see

$$y_p' = Ay_1 + By_2 + t[Ay_1' + By_2'] \quad \& \quad y_p'' = 2(Ay_1' + By_2') + t[Ay_1'' + By_2'']$$

If we substitute this into the ODE, we see

$$2(Ay_1' + By_2') = F\sin(\omega_0 t) \implies A = -\frac{F}{2\omega_0} \quad \& \quad B = 0$$

Thus the general solution to the ODE is

$$\boxed{y = y_{hom} + y_p = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) - \frac{Ft}{2\omega_0}\cos(\omega_0 t)}$$

The IVP may be solved now, we see that

$$\boxed{y = y_0\cos(\omega_0 t) + \frac{F + 2\omega_0 v_0}{2\omega_0^2}\sin(\omega_0 t) - \frac{Ft}{2\omega_0}\cos(\omega_0 t)}$$

Quiz A helical spring has a period of 4 sec when a 16-lb weight is attached. If the 16-lb weight is removed and a weight W is attached, the spring oscillates with a period of 3 sec. Find the weight W.

Solution Since $my'' + ky = 0$ in an helical spring and we define the period of the oscillator to be T such that $f(t+T) = f(t)$ for all $t \in \mathbb{R}$. The solutions take the form $y = \cos\left(\sqrt{\frac{k}{m}}t + \delta\right)$, and $\cos x$ has a period of 2π, thus

$$\sqrt{\frac{k}{m_i}}T_i = 2\pi \implies m_i = \frac{kT_i^2}{4\pi^2}$$

using the data given we see

$$k = \frac{4\pi^2 m_1}{T_1^2} = 4\pi^2 \frac{lb}{s^2}$$

Thus

$$m_2 = \frac{kT_2^2}{4\pi^2} = T_2^2 = 9lb$$

□