Tutorial Problems #4

MAT 267 – Advanced Ordinary Differential Equations – Winter 2016

Christopher J. Adkins

Solutions

n-th Order Linear Differential Equations with Constant Coefficients Always try $e^{\lambda x}$ as a solution to

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y^{(0)} = 0$$

since

$$\frac{d^n}{dx^n} e^{\lambda x} = \lambda^n e^{\lambda x} \implies e^{\lambda x} \left(a_n \lambda^n + a_{n-1} \lambda^{n-1} \ldots + a_0 \right) = 0$$

Thus $e^{\lambda x}$ is a solution if (since the exponential is never zero)

$$P(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} \ldots + a_0 = 0$$

i.e. λ is a root of $P(\lambda)$, which is called the characteristic polynomial. By the fundamental theorem of algebra, we know that we'll always find n roots over the complex numbers. Thus we've found n solutions to the ODE

Why Is It Called The Characteristic Equation? Recall from last time, we saw

$$ay'' + by' + cy = 0 \iff \dot{x} = \underbrace{\frac{1}{a} \begin{pmatrix} 0 & a \\ -c & -b \end{pmatrix}}_{=A} x \quad \text{where} \quad x = \begin{pmatrix} y \\ y' \end{pmatrix}$$

Notice that the characteristic equation, the one that determines the eigenvalues, is the same as the previous polynomial,

$$P(\lambda) = \det(A - 1\lambda) = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0 \implies a\lambda^2 + b\lambda + c = 0$$

Repeated Roots On can check that you have repeated roots from your characteristic equation. To form a full basis for your solution space, i.e. the fundamental solutions, you can just stick t in front of the exponential for every solution you're missing. i.e. if $P(\lambda) = (\lambda - a)^3$, we'd have

$$y_1 = e^{at} \quad \& \quad y_2 = te^{at} \quad \& \quad y_3 = t^2 e^{at}$$

pg.220 - #17 Solve

$$y'' - 2ay' + a^2 y = 0$$

Solution The characteristic polynomial for the ODE is

$$P(\lambda) = \lambda^2 - 2a\lambda + a^2 = (\lambda - a)^2$$

i.e. we have a repeated root of $\lambda = a$. Let's try $y(t) = te^{at}$ as a solution,

$$2a\underbrace{\left(e^{at} + a^2te^{at}\right)}_{y''} - 2a\underbrace{\left(e^{at} + ate^{at}\right)}_{y'} + a^2\underbrace{\left(te^{at}\right)}_{y} = 0$$

Thus the general solution is given by

$$y(t) = c_1e^{at} + c_2te^{at}$$

□

Example Solve the IVP

$$3y''' + 5y'' + y' - y = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = -1$$

Solution The characteristic equation for the ODE is

$$P(\lambda) = 3\lambda^3 + 5\lambda^2 + \lambda - 1 = (\lambda + 1)^2(3\lambda - 1) = 0 \implies \lambda = \frac{1}{3} \quad \& \quad \lambda = -1(\text{ Repeated})$$

Thus general solution is given by

$$\boxed{y(t) = c_1e^{-t} + c_2te^{-t} + c_3e^{t/3}}$$

The initial data implies

$$\begin{cases} c_1 + c_3 = 0 \\ -c_1 + c_2 + c_3/3 = 1 \\ c_1 + 2c_2 + c_3/9 = -1 \end{cases} \implies c_1 = -\frac{9}{16}, c_2 = \frac{1}{4}, c_3 = \frac{9}{16}$$

Hence the solution to the IVP is

$$\boxed{y(t) = \frac{9}{16}e^{t/3} + \left(\frac{t}{4} - \frac{9}{16}\right)e^{-t}}$$

□

Complex Eigenvalues In the case of complex eigenvalues, it seems we have a complex valued solution...though using Euler's Identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we may rewrite the solution in terms of real valued functions. Let $\lambda_+ = a + bi$ and $\lambda_- = a - bi$, then

$$\begin{aligned} y(t) = Ce^{\lambda_+ t} + \bar{C}e^{\lambda_- t} =& e^{at}(Ce^{ibt} + \bar{C}e^{-ibt}) \\ =& e^{at}((C + \bar{C})\cos(bt) + i(C - \bar{C})\sin(bt) \\ =& e^{at}(c_1\cos(bt) + c_2\sin(bt)) \end{aligned}$$

Euler Equations Consider

$$ax^2y'' + bxy' + cy = 0 \quad x > 0$$

At first glance it seems like a foreign equation, but let's apply the change of variables $x \to z = \ln(x)$ to the ODE. We see via chain rule that

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \implies y' = \frac{1}{x}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

Thus, the ODE in z becomes

$$a\frac{d^2y}{dz^2} + (b-a)\frac{dy}{dz} + cy = 0$$

i.e. an ODE with constant coefficients. We know the solutions take exponential form...or in terms of the variable x we have

$$e^{\lambda z} = e^{\lambda \ln(x)} = e^{\ln(x^\lambda)} = x^\lambda$$

Thus we see trying $y(x) = x^\lambda$ will amount to the same ole story.

HW - #1 Consider the equation

$$y^{(n)} + p_{n-1}y^{(n-1)} + \ldots + p_1y' + p_0y = 0$$

with $p_j(x)$ for each $j \in [0, n-1]$ continuous on $[a, b]$. Suppose that y is a solution with infinitely many zeros in the interval $[a_1, b_1]$ such that $a < a_1 < b_1 < b$. Prove that $y \equiv 0$ on (a, b).

Proof By the Bolzano Weierstrass Theorem, we know the infinite sequence of zeros $\{x_m\}_{m=1}^{\infty}$ has a converging subsequence to $x_0 \in [a_1, b_1]$. By definition of a solution, we require y and it's derivatives to be continuous on $[a, b]$. This leads us to consider the neighbourhood $B_\delta(x_0) = \{x : x \in (x_0 - \delta, x_0 + \delta)\}$ for any $\delta > 0$, and check if $y \neq 0$ there. This will only happen if $y'(x_0) \neq 0$, but $B_\delta(x_0)$ has infinitely many zero's of y, so this means that $y' = 0$ on $B_\delta(x_0)$. Repeating this argument with y', and moving up to higher derivatives, we conclude

$$y(x_0) = y'(x_0) = \ldots = y^{n-1}(x_0) = 0$$

Next we'll show that the trivial solution is the only solution to a null data problem. Define

$$\xi = \sum_{k=0}^{n-1}(y^{(k)})^2 \geqslant 0$$

Then the derivative will give us

$$\xi' = 2\sum_{k=0}^{n-1} y^{(k)}y^{(k+1)}$$

Plug the ODE into the above

$$\xi' = yy' + \ldots + y^{(n-2)}y^{(n-1)} + y^{(n-1)}(-p_{n-1}y^{(n-1)} - \ldots p_0y)$$

Using the Inequality $2ab \leqslant a^2 + b^2$ and the fact that p_j for each j is continuous (we may bound each above), we see

$$\xi' \leqslant K\xi$$

for some constant K. Gronwall's Inequality now tells us

$$\xi(x) \leqslant \xi(x_0)e^{K(x-x_0)} = 0, \quad x \geqslant x_0$$

A similar bound is found for $\xi' \geqslant -K\xi$ (use $2ab \geqslant -a^2 - b^2$) to conclude that $\xi = 0$ for $x < x_0$, together these imply that $y \equiv 0$ on $[a, b]$. □

Quiz Suppose that vector functions $y^1(x), \ldots, y^n(x)$ taking values in $\mathbb{R}^n$ are linearly independent on the interval $[a, b]$, and all their coordinates are differentiable on $[a, b]$. Show that there exists a matrix function $A(x) = (a_{ij}(x))_{1\leqslant i,j\leqslant n}$ such that $y^1(x), \ldots, y^n(x)$ are solution of the system $y' = Ay$ on $[a, b]$.

Solution Define the matrix X to have columns y_j:

$$X(x) = (y^1, \ldots, y^n)$$

Then we'd like this matrix to solve

$$X' = AX$$

So we now have an equation for the $A(x)$ we'd like to construct, we simply set

$$X'(x)X^{-1}(x) = A(x)$$

Note the inverse is well defined since all vectors are linearly independent. This is such an A. □