Tutorial Problems #2

MAT 267 – Advanced Ordinary Differential Equations – Fall 2014

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Solutions

pg.90 - # 7 Solve

$$(x^4y^2 - y)dx + (x^2y^4 - x)dy = 0$$

Solution Notice the symmetry, so lets check if the equation is exact. Let $M = x^4y^2 - y$ and $N = x^2y^4 - x$, then

$$M_y = 2x^4y - 1 \quad \& \quad N_x = 2xy^4 - 1$$

i.e. it's not exact, but we see

$$N_x - M_y = 2xy(y^3 - x^3) \quad \& \quad xM - yN = -x^2y^2(y^3 - x^3)$$

In a previous exercise we saw that

$$\mu(xy) = \exp\left(\int \frac{N_x - M_y}{xM - yN} d(xy)\right) = \exp\left(-2\int \frac{d(xy)}{xy}\right) = \exp -2\ln|xy| = \frac{1}{x^2y^2}$$

works as an integrating factor provide the function $N_x - M_y/xM - yN$ depended on xy, which in our case does! Thus the ODE becomes

$$\underbrace{\left(x^2 - \frac{1}{x^2y}\right)}_{=\tilde{M}} dx + \underbrace{\left(y^2 - \frac{1}{xy^2}\right)}_{=\tilde{N}} dy = 0$$

after multiplying by our integrating factor. It's easily seen that the ODE is now exact, so we integrate the components as usual.

$$\begin{aligned} F(x,y) &= \int \tilde{M} dx \oplus \int \tilde{N} dy \\ &= \int \left(x^2 - \frac{1}{x^2y}\right) dx \oplus \int \left(y^2 - \frac{1}{xy^2}\right) dy \\ &= \frac{x^3}{3} + \frac{1}{xy} \oplus \frac{y^3}{3} + \frac{1}{xy} \\ &= \frac{x^3 + y^3}{3} + \frac{1}{xy} \end{aligned}$$

Thus the general solution is

$$\boxed{\frac{x^3 + y^3}{3} + \frac{1}{xy} = C}$$

pg.90 - # 9 Solve

$$\underbrace{\left(\arctan(xy) + \frac{xy - 2xy^2}{1 + x^2y^2}\right)}_{M} dx + \underbrace{\frac{x^2 - 2x^2y}{1 + x^2y^2}}_{N} dy = 0$$

Solution We check if the equation is exact.

$$M_y = \frac{2x - 4xy}{1 + x^2y^2} - \frac{2x^3y^2 - 4x^3y^3}{(1 + x^2y^2)^2} = N_x$$

Since the equation is exact, we may integrate the components and take the linearity independent parts.

$$\begin{aligned} F(x, y) &= \int M dx \oplus \int N dy \\ &= x \arctan(xy) - \log(x^2y^2 + 1) \oplus x \arctan(xy) - \log(x^2y^2 + 1) \\ &= x \arctan(xy) - \log(x^2y^2 + 1) \end{aligned}$$

Thus the general solution is

$$\boxed{x \arctan(xy) - \log(x^2y^2 + 1) = C}$$

pg.103 - # 5 Solve

$$y' \sin y + \sin x \cos y = \sin x$$

Solution Notice if $z = \cos y$, then $z' = -y' \sin y$. Thus we're able to rewrite the ODE as

$$z' \underbrace{- \sin x}_{=p} z = \underbrace{- \sin x}_{=g}$$

In this form the ODE is first order linear. We know the solution is given by

$$z(x) = \frac{1}{\mu(x)} \int g(x)\mu(x)dx \quad \text{where} \quad \mu(x) = \exp\left(\int p(x)dx\right) = \exp\left(-\int \sin x dx\right) = \exp(\cos x)$$

Thus

$$z(x) = e^{-\cos x} \int - \sin x e^{\cos x} dx = e^{-\cos x}(e^{\cos x} + C) = 1 + Ce^{-\cos x}$$

In terms of the original function, we have

$$\cos(y) = 1 + Ce^{-\cos x} \implies \boxed{y(x) = \arccos(1 + Ce^{-\cos x})}$$

□

Riccati Equation Consider the ODE

$$y' = f(x) + g(x)y + h(x)y^2, \quad h(x) \neq 0$$

If y_1 is a particular solution of this equation, show that the substitution

$$y = y_1 + \frac{1}{u}, \quad y' = y_1' - \frac{1}{u^2}u'$$

will transform the equation into the first order linear

$$u' + (g + 2hy_1)u = -h$$

Solution Using the change of variables suggested, we see

$$\begin{aligned} y' = f(x) + g(x)y + h(x)y^2 &\implies y_1' - \frac{1}{u^2}u' = f(x) + g(x)\left(y_1 + \frac{1}{u}\right) + h(x)\left(y_1 + \frac{1}{u}\right)^2 \\ &\implies u^2y_1' - u' = u^2(f(x) + g(x)y_1 + h(x)y_1^2) + ug(x) + h(x) + 2y_1h(x)u \\ &\implies -u' = (g(x) + 2hy_1)u + h \end{aligned}$$

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$$y' = \frac{1}{x^2} - \frac{y}{x} - y^2, \quad y_1(x) = \frac{1}{x}$$

Solution This is an Riccati Equation, so we may use the suggested change of variables

$$y = y_1 + \frac{1}{u}$$

We see that $f(x) = \frac{1}{x^2}, g(x) = -\frac{1}{x}$ and $h(x) = -1$, thus the resulting equation will be

$$u' + (g + 2hy_1)u = -h \implies u' + \frac{3}{x}u = 1$$

Now the equation is first order linear, we see a nice integrating factor of $1/x^3$ will do the job. Thus

$$u(x) = x^3\int \frac{1}{x^3}dx = \frac{-x + Cx^3}{2} + \implies \boxed{y(x) = \frac{1}{x} + \frac{2}{-x + Cx^3}}$$

□

Infinitely Many Solutions Suppose the domain D is a strip $[a, b] \times \mathbb{R}$, and let $f(x, y)$ be continuous and bounded on D. It is possible that more than one integral curve of the equation $\frac{dy}{dx} = f(x, y)$ passes through a given point (x_0, y_0) inside the strip, $a < x_0 < b$. Prove that there are two integral curves $y = \varphi_1(x)$ and $y = \varphi_2(x)$ of this equation, the maximum and minimum solutions such that:

- $\varphi_1(x_0) = \varphi_2(x_0) = y_0 \quad \& \quad \varphi_2(x) \leqslant \varphi_1(x) \quad \forall x \in [a, b]$
- The entire part of the strip between $\varphi_2(x)$ and $\varphi_1(x)$ can be completely filled by integral curves passing through (x_0, y_0).
- There are no integral curves passing through (x_0, y_0) which lie outside of this part of the strip.

Solution Here's a sketch. We'll construct both the max and min solution using approximation lines. We know by the fundamental theorem of calculus that

$$y' = f(x, y) \quad y(x_0) = y_0 \iff y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds$$

By Peano's existence theorem, we know there exists at least 1 integral curve through (x_0, y_0) since f is continuous and bounded (gives a uniformly convergent subsequence). Call y_{max} the largest integral curve and y_{min} the smallest. Then we have that

$$y_{max} - y_{min} = \int_{x_0}^{x} [f(s, y_{max}(s)) - f(s, y_{min}(s))]ds \geqslant 0$$

These may be constructed using an Euler Approximation on the integral to recursively build the max and min (or use lines and the differential). Any curve in-between may be also created since we may take the any value between the min and max in the recursion.

Quiz Solve

$$(x - \sin y)dy + \tan y dx = 0, \quad y(1) = \pi/6$$

Solution Notice we may write this into a first order linear equation for x:

$$\frac{dx}{dy} = \frac{\sin y - x}{\tan y} = \cos y - \frac{x}{\tan y} \implies x' + \frac{1}{\tan y} x = \cos y$$

Thus with integrating factor

$$\mu(y) = \exp\left[\int \frac{dy}{\tan y}\right] = \exp \ln \sin y = \sin y$$

Then we know

$$x(y) = \frac{1}{\mu(y)} \int \mu(y) g(y) dy = \frac{1}{\sin y} \int \cos y \sin y dy = \frac{\sin y}{2} + \frac{C}{\sin y} \quad C \in \mathbb{R}$$

The initial data implies the constant must be given by

$$1 = x(\pi/6) = \frac{1}{4} + 2C \implies C = \frac{3}{8} \implies \boxed{x(y) = \frac{\sin y}{2} + \frac{3}{8 \sin y}}$$

or

$$\boxed{8x \sin y = 4 \sin^2 y + 3}$$

□