Tutorial Problems #1

MAT 267 – Advanced Ordinary Differential Equations – Winter 2016

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Solutions

pg.56-# 19 Solve

$$\begin{cases} (1-x)dy = x(y+1)dx \\ y(0) = 0 \end{cases}$$

Solution The equation is separable, thus

$$\frac{dy}{y+1} = \frac{xdx}{1-x} \implies \int \frac{dy}{y+1} = \int \frac{xdx}{1-x} \implies \ln|y+1| = \ln\left|\frac{1}{1-x}\right| + x + C \implies y(x) = \frac{\tilde{C}e^x}{1-x} - 1$$

is the general solution. The initial condition fixes the constant.

$$y(0) = 0 \implies 0 = \tilde{C} - 1 \implies \tilde{C} = 1$$

Thus the solution to the IVP is

$$\boxed{y(x) = \frac{e^x}{1-x} - 1}$$

□

pg.69 - #6 Solve

$$(x+y)dx + (2x+2y-1)dy = 0$$

Solution We see the lines in the equation are parallel since

$$u = x + y \implies 2u - 1 = 2x + 2y - 1$$

Thus we make the change of variables u as above, we see $du = dx + dy$ and

$$(1-u)dx + (2u-1)du = 0$$

is a separable equation. Thus the implicit solution is given by

$$\int dx = \int \frac{1-2u}{1-u}du = \int \frac{1}{1-u} + \int \left(2 - \frac{2}{1-u}\right) du \implies x + C = \ln|1-u| + 2u - 2\ln|1-u|$$

$$\implies \boxed{x + 2y - \ln|x+y-1| = C}$$

□

pg.69 - #10 Solve

$$(3x - 2y + 4)dx - (2x + 7y - 1)dy = 0$$

Solution We know it is possible to convert this to a homogeneous equation, we just need to find the change of variables. We'll use the differential method, i.e. we want u and v such that

$$3x - 2y + 4 = u \quad \& \quad -2x - 7y + 1 = v$$

This implies that

$$\begin{cases} du = 3dx - 2dy \\ dv = -2dx - 7dy \end{cases} \iff \begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -2 & -7 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

By inverting the matrix we see that

$$\frac{1}{25} \begin{pmatrix} 7 & -2 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Thus the ODE becomes the following in coordinates u and v,

$$\begin{aligned} udx + vdy =& u\left(\frac{7}{25}du - \frac{2}{25}\right) + v\left(-\frac{2}{25}du - \frac{3}{25}dv\right) \\ =& \frac{1}{25}[(7u - 2v)du + (-2u - 3v)dv] \end{aligned}$$

Now the equation is homogeneous, which we know is possible to solve using $u = tv$, $du = tdv + vdt$. Substitute the change of variables again. (note we've dropped the 1/25 since the RHS is 0)

$$(7tv - 2v)(tdv + vdt) + (-2tv - 3v)dv = (7t^2 - 4t - 3)vdv + (7t - 2)v^2dt = 0$$

This equation is separable, and we see the solution is (using $\omega = 7t^2 - 4t - 3, d\omega = 2(7t - 2)dt$)

$$\int \frac{dv}{v} = \int \frac{(2 - 7t)dt}{7t^2 - 4t - 3} = -\frac{1}{2}\int \frac{d\omega}{\omega} \implies \ln|v^2\omega| = C$$

Now we just back substitute everything to revert to the original coordinates.

$$\omega v^2 = \tilde{C} \implies 7(vt)^2 - 4tv^2 - 3v^2 = \tilde{C} \implies 7u^2 - 4uv - 3v^2 = \tilde{C}$$

Hence the implicit general solution is

$$\boxed{7y^2 + (2 - 4x)y + 3x^2 + 8x = Const}$$

□

pg.79 - #16 Solve

$$\begin{cases} \sin x \cos y dx + \cos x \sin y dy = 0 \\ y(\pi/4) = \pi/4 \end{cases}$$

Solution By the symmetry of the equation, we check if it is exact. i.e

$$M_y = N_x \quad \text{where} \quad \underbrace{\sin x \cos y}_{=M}\, dx + \underbrace{\cos x \sin y}_{=N}\, dy = 0$$

Clearly

$$M_y = -\sin x \sin y = N_x$$

Thus the equation is exact, the solution is therefore given by a level set of linearity independent factors of the integrated functions. I write this as

$$\begin{aligned} F(x,y) &= \int M dx \oplus \int N dy \\ &= \int \sin x \cos y dx \oplus \int \cos x \sin y dy \\ &= -\cos x \cos y \oplus -\cos x \cos y \\ &= -\cos x \cos y \end{aligned}$$

Thus the general solution is

$$\cos x \cos y = C$$

The initial data implies that

$$C = \cos(\pi/4)\cos(\pi/4) = \frac{1}{2} \implies \boxed{1 = 2\cos x \cos y}$$

is the implicit solution to the IVP. □

pg.79 - #12 Solve

$$x\sqrt{x^2+y^2}dx - \frac{x^2 y}{y - \sqrt{x^2+y^2}}dy = 0$$

Solution Notice we may rewrite the 2nd component since

$$-\frac{x^2 y}{y-\sqrt{x^2+y^2}} = -\frac{x^2 y}{y-\sqrt{x^2+y^2}}\frac{y+\sqrt{x^2+y^2}}{y+\sqrt{x^2+y^2}} = -\frac{x^2y^2 + x^2 y\sqrt{x^2+y^2}}{y^2-x^2-y^2} = y^2 + y\sqrt{x^2+y^2}$$

It's easy to check that the equation is exact since

$$M_y = \frac{xy}{\sqrt{x^2+y^2}} = N_x$$

Thus solution is given by

$$\begin{aligned} F(x,y) &= \int M dx \oplus \int N dy \\ &= \int x\sqrt{x^2+y^2}dx \oplus \int (y^2 + y\sqrt{x^2+y^2})dy \\ &= \frac{1}{3}(x^2+y^2)^{3/2} \oplus \frac{y^3}{3} + \frac{1}{3}(x^2+y^2)^{3/2} \\ &= \frac{(x^2+y^2)^{3/2} + y^3}{3} \end{aligned}$$

Thus the implicit solution is given by

$$\boxed{(x^2+y^2)^{3/2} + y^3 = const}$$

□

Quiz Solve

$$ydy + xdx = 3xy^2dx, \quad y(2) = 1$$

Solution Rewrite the ODE to

$$ydy = x(3y^2 - 1)dx$$

Clearly this equation is separable, thus the implicit solution is given by

$$\int \frac{y}{3y^2-1}dy = \int xdx \implies \frac{1}{6}\ln|3y^2-1| = \frac{x^2}{2} + C \implies 3y^2 - 1 = \tilde{C}e^{3x^2} \implies y = \pm\sqrt{\bar{C}e^{3x^2} + \frac{1}{3}}$$

The initial data implies that we need the + sign, and

$$\bar{C} = \frac{2e^{-12}}{3} \implies \boxed{y = \frac{1}{\sqrt{3}}\sqrt{2e^{3(x^2-4)}+1}}$$

□