Tutorial Problems #9

MAT 267 – Advanced Ordinary Differential Equations – Fall 2014

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SOLUTIONS

Exercise from Section 24 Draw precisely the behaviour of solutions of

$$\frac{dy}{dx} = \frac{3x+1}{x-3y}$$

Solution Rewrite this in terms of a matrix system, namely

$$\dot{z} = \underbrace{\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}}_{=A} z \quad \text{where} \quad z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

To graph the system, lets solve it! First we find the eigenvalues of A:

$$P(\lambda) = \det(A - I\lambda) = \begin{vmatrix} 1-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 10$$

Clearly the roots of the above equation are

$$\lambda_\pm = 1 \pm 3i$$

Next up we have the eigenvectors, these are easily found by looking at the kernel:

$$\ker(A - I\lambda) = \ker \begin{pmatrix} 3i & -3 \\ 3 & 3i \end{pmatrix} = \operatorname{span} \begin{pmatrix} i \\ -1 \end{pmatrix} \implies \vec{\lambda}_- = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

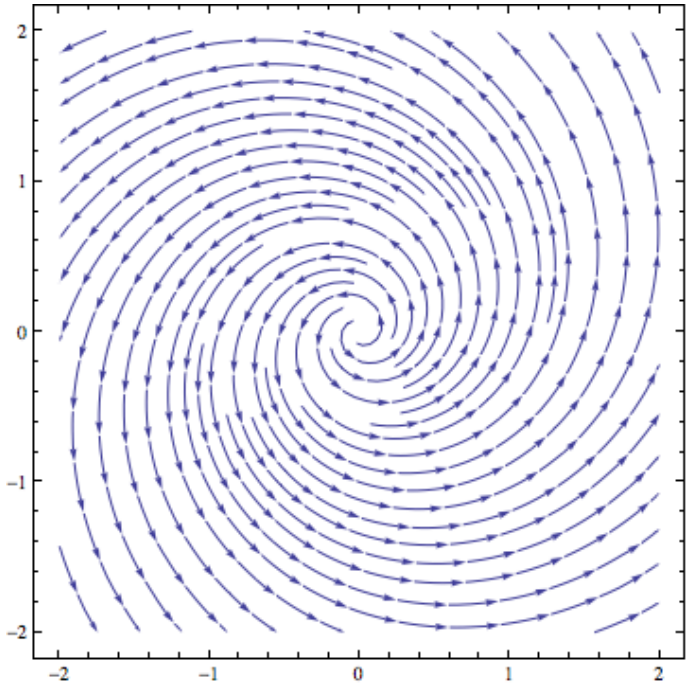
Since the eigenvalues are complex, we immediately have the other by complex conjugation (since A is real)

$$\vec{\lambda}_+ = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Thus the solution to the system is

$$z(t) = C_1 \vec{\lambda}_+ e^{\lambda_+ t} + C_2 \vec{\lambda}_- e^{\lambda_- t} = e^t \left(\tilde{C}_1 \begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix} + \tilde{C}_2 \begin{pmatrix} -\sin(3t) \\ \cos(3t) \end{pmatrix} \right)$$

Hence, we obviously have that the phase portrait corresponding to this system is



Exercise from Section 25 Construct a system of the type

$$\frac{dx}{dt} = x(a + bx + cy)$$
$$\frac{dy}{dt} = y(d + ex + fy)$$

such that $(1, 1)$ is a critical point and its linearization is a node of type II (i.e. an improper node). Note this is the general form for the competitive Lotka-Volterra equations. The region of importance is $x, y \geqslant 0$.

Solution We require $(1, 1)$ to be a critical point (i.e. $\dot{z} = 0$ when $z_0 = (1, 1)$). Therefore we like to find constants such that the linear transform $\tilde{z} = z - z_0$, linearizes the system. i.e.

$$0 = \begin{pmatrix} a + b + c \\ d + f + e \end{pmatrix}$$

Clearly we need

$$\boxed{a + b + c = 0} \quad \& \quad \boxed{d + e + f = 0}$$

for $(1, 1)$ to be a critical point. Now that we have that out of the way, we'll fix the requirement of the system falling into an improper node (Type II). Perform the change of variables:

$$\frac{dx}{dt} = x(a + bx + cy) \to \frac{d\tilde{x}}{dt} = (\tilde{x} + 1)(a + b(\tilde{x} + 1) + c(\tilde{y} + 1)) = \underbrace{a + b + c}_{=0} + \underbrace{(a + 2b + c)x + cy}_{linear} + \underbrace{bx^2 + cxy}_{nonlinear}$$

$$\frac{dy}{dt} = y(d + ex + fy) \to \frac{d\tilde{y}}{dt} = (\tilde{y} + 1)(d + e(\tilde{x} + 1) + f(\tilde{y} + 1)) = \underbrace{d + e + f}_{=0} + \underbrace{ex + (d + 2f + e)y}_{linear} + \underbrace{fy^2 + exy}_{nonlinear}$$

In terms of matrices, the linearized system given by

$$\dot{\tilde{z}} = \begin{pmatrix} a + 2b + c & c \\ e & d + 2f + e \end{pmatrix} \tilde{z} = \begin{pmatrix} b & c \\ e & f \end{pmatrix} \tilde{z}$$

We need a repeated root in the characteristic equation to have a improper node, plug and chug!

$$P(\lambda) = \det(A - I\lambda) = (b-\lambda)(f-\lambda) - ec = \lambda^2 - (f+b)\lambda + fb - ec$$

If we plug this into the quadratic formula we see

$$\lambda_{\pm} = \frac{f+b \pm \sqrt{(f+b)^2 - 4(fb-ec)}}{2} = \frac{f+b \pm \sqrt{(f-b)^2 + 4ec}}{2}$$

Thus, we see our third condition here. Namely

$$\boxed{(f-b)^2 + 4ec = 0}$$

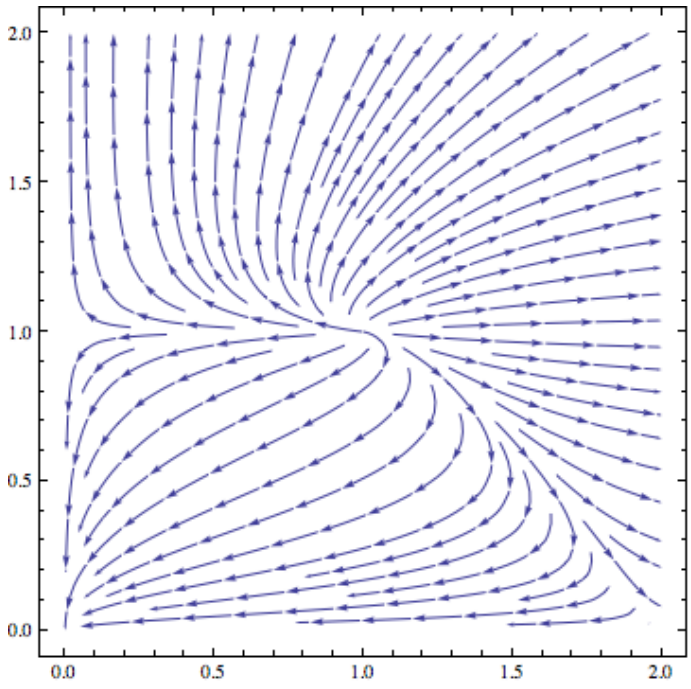
The 3 conditions we derived are sufficient to deduce that the linearized system is an improper node. As an example, take

$$a = -4 \quad \& \quad d = -2 \quad \& \quad e = 0 \quad \& \quad b = f = c = 2$$

i.e.

$$\boxed{\begin{pmatrix} x' \\ y' \end{pmatrix} = -\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2x^2 + 2xy \\ 2y^2 \end{pmatrix}}$$

The phase portrait around $(1,1)$ looks as follows



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Exercise from Section 26 Consider the system

$$\frac{dx}{dt} = x(1 - x - 2y)$$

$$\frac{dy}{dt} = y(1 - 2x - y)$$

Analyze critical points $(0,0), (0,1)$ and $(1,0)$ in the above.

Solution Lets start with what the corresponding linearized equations for each critical point. Clearly $z_0 = (0,0)$ has system

$$\boxed{\dot{z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z \implies \text{node}}$$

The critical point $z_0 = (0,1)$ corresponds to $\tilde{x} = x$ and $\tilde{y} = y - 1$, hence

$$\frac{dx}{dt} = x(1 - x - 2y) \to \frac{d\tilde{x}}{dt} = \tilde{x}(1 - \tilde{x} - 2(\tilde{y} + 1)) = -\tilde{x} - \tilde{x}^2 - 2\tilde{x}\tilde{y}$$

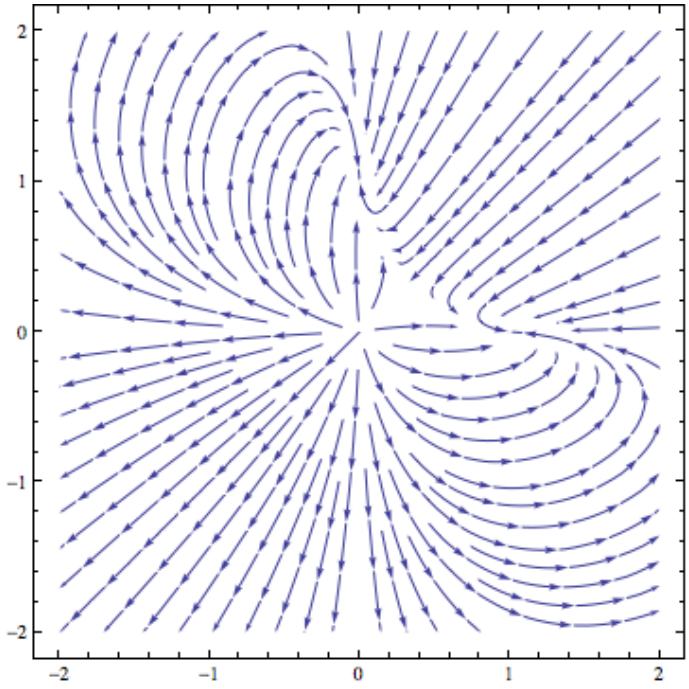
$$\frac{dy}{dt} = y(1 - 2x - y) \to \frac{d\tilde{y}}{dt} = (\tilde{y} + 1)(1 - 2\tilde{x} - (\tilde{y} + 1)) = \tilde{y}(-1 - 2\tilde{x} - \tilde{y}) - 2\tilde{x}$$

$$\implies \boxed{\dot{\tilde{z}} = -\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \tilde{z} \implies \text{improper node}}$$

The critical point $z_0 = (1,0)$ corresponds to $\tilde{x} = x - 1$ and $\tilde{y} = y$, but by symmetry i.e. $x \to y$ and $y \to x$, we know that the system is

$$\boxed{\dot{\tilde{z}} = -\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tilde{z} \implies \text{improper node}}$$

This fully characterizes the system. By finding the eigenvectors (generalized eigenvectors) for each of the three cases, we can easily see the portrait is



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Exercise from Section 27 Draw the phase portrait and describe solutions precisely for the system

$$\frac{dx}{dt} = x(1 - y)$$

$$\frac{dy}{dt} = y \cdot \min(1, x - 1)$$

Solution The min function breaks us into two cases. Namely $x > 2$ and $x \leqslant 2$, i.e

$$\frac{dy}{dt} = y \quad \& \quad \frac{dy}{dt} = y(x-1)$$

respectively. Clearly the critical points of the system are

$$z_0 = (0,0) \quad \& \quad z_0 = (1,1)$$

Linearizing about $z_0 = (0,0)$ gives

$$\dot{z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z \implies \text{saddle}$$

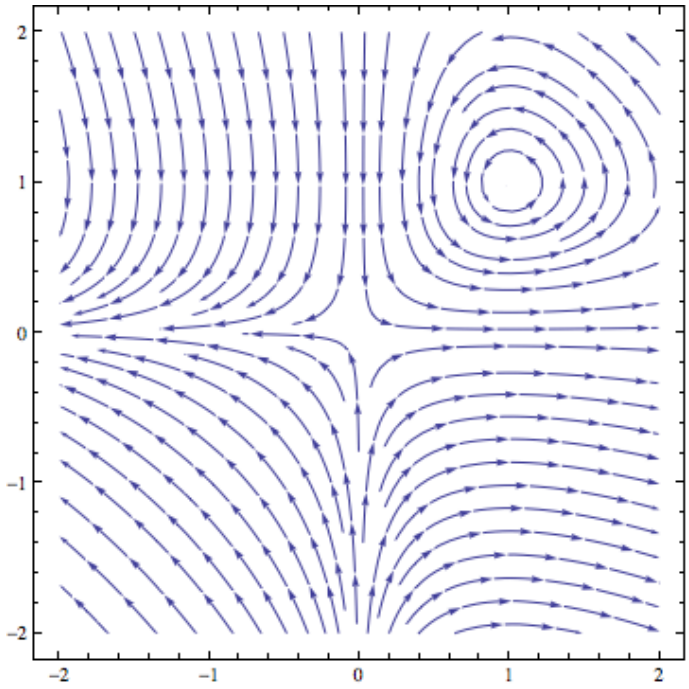
clearly the eigenvectors are along the axis lines. we know how to draw this, so lets move on to the second critical point. Linearizing about $z_0 = (1,1)$ can be done with our change of variables $\tilde{x} + 1 = x$ and $\tilde{y} + 1 = y$ as previously, we see

$$\dot{\tilde{z}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{z}$$

We cannot read the eigenvalues directly off this system, but its clear that the characteristic equation is

$$P(\lambda) = \lambda^2 + 1 = (\lambda - i)(\lambda + i) = 0$$

Thus, the system is centre. Note that we see the rotation is counter clockwise since the top right entry is negative. Now its easy to draw the system, it's phase portrait is



Saddle Example Solve the following system, draw direction field and a phase portrait. Describe the behaviour of the solutions as $t \to \infty$

$$\mathbf{x}' = \underbrace{\frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}}_{A} \mathbf{x}$$

Solution By now we know the solution is completely characterized by the eigenvalues and eigenvectors of the above matrix. To make the computation nicer, recall that the eigenvalues of A are 4 times what we actually want. Now, let's compute the characteristics equation to find the eigenvalues of A.

$$P(\lambda) = \det(A - I\lambda) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (\lambda - 8)(\lambda - 2) = 0 \implies \lambda_1 = 8 \,\&\, \lambda_2 = 2$$

Now that we've found the eigenvalues, we must find the eigenvectors! They are easily computed by looking at the kernel of the map evaluated at the eigenvalues

$$\ker(A - I\lambda_1) = \ker \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies \vec{\lambda}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\ker(A - I\lambda_2) = \ker \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies \vec{\lambda}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

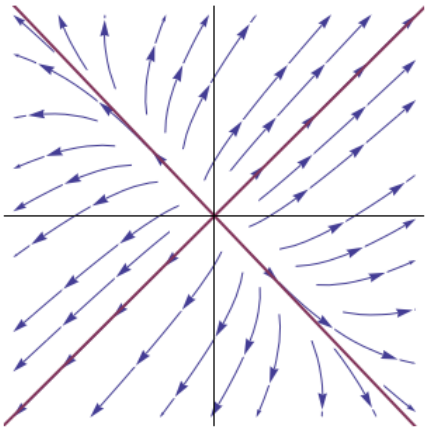
Since eigenvectors are invariant under scaling, we therefore have that actual eigenvalues and eigenvectors are

$$\lambda_1 = 2 \quad \& \quad \vec{\lambda}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \& \quad \lambda_2 = \frac{1}{2} \quad \& \quad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Thus the solution is

$$\boxed{\mathbf{x}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2}}$$

where $C_1, C_2 \in \mathbb{R}$. The system looks like



Example of drawing Consider $\mathbf{x}' = A\mathbf{x}$. If given the eigenvectors and eigenvalues:

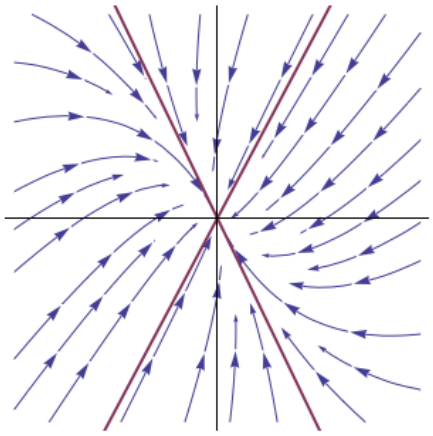
(a) Sketch a phase portrait of the system.

(b) Sketch the trajectory passing through the initial point (2,3)

 # 1

$$\lambda_1 = -1 \quad \vec{\lambda}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \& \quad \lambda_2 = -2 \quad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies \boxed{x(t) = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}}$$

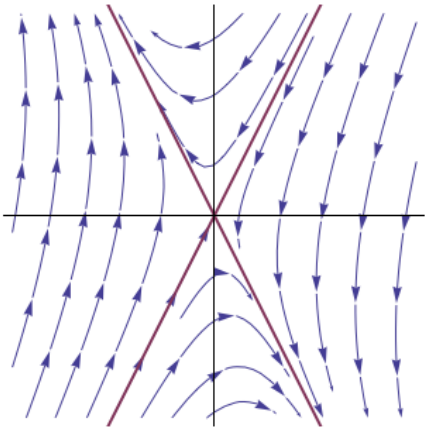
with a portrait like



 # 2

$$\lambda_1 = 1 \quad \vec{\lambda}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \& \quad \lambda_2 = -2 \quad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies \boxed{x(t) = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}}$$

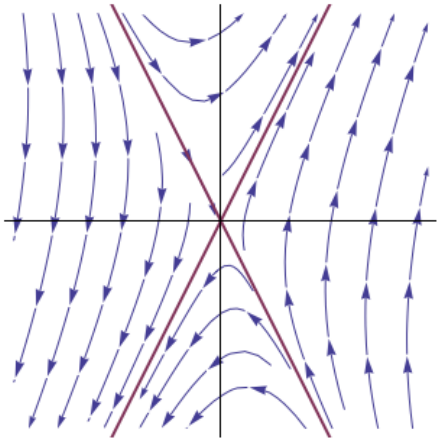
with a portrait like



 # 3

$$\lambda_1 = -1 \quad \vec{\lambda}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \& \quad \lambda_2 = 2 \quad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies \boxed{x(t) = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}}$$

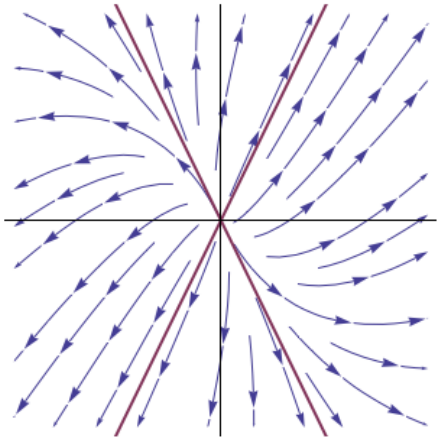
with a portrait like



 # 4

$$\lambda_1 = 1 \quad \vec{\lambda}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \& \quad \lambda_2 = 2 \quad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies \boxed{x(t) = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}}$$

with a portrait like



Quiz Question Consider the system

$$\frac{dx}{dt} = x(1 - 2x - y)$$
$$\frac{dy}{dt} = y(1 - x - 2y)$$

Draw the phase portrait and analyze the critical point $(1/3, 1/3)$.

Solution Let's linearize around the critical point. Clearly the Jacobian is given by

$$\frac{\partial x_i'}{\partial x_j} = \begin{pmatrix} 1 - 4x - y & -x \\ -y & 1 - x - 4y \end{pmatrix}$$

Thus the linearized system around $(1/3, 1/3)$ is given by

$$\dot{z} = -\frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} z$$

The constant doesn't affect the type of eigenvalues, only their values. So we compute

$$P(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 1 \implies \lambda_\pm = 2 \pm \sqrt{3}$$

Hence this critical point is a stable node locally (don't forget the minus out front). To draw it, let's find the eigenvectors. We see

$$\lambda_+ \implies \ker \begin{pmatrix} -\sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \implies \vec{\lambda}_+ = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$$

$$\lambda_- \implies \ker \begin{pmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \implies \vec{\lambda}_+ = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$

Since λ_+ dominates for large t, we have that