Tutorial Problems #4

MAT 267 – Advanced Ordinary Differential Equations – Fall 2014

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n-th Order Linear Differential Equations with Constant Coefficients   Always try $e^{\lambda x}$ as a solution to

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y^{(0)} = 0$$

since

$$\frac{d^n}{dx^n} e^{\lambda x} = \lambda^n e^{\lambda x} \implies e^{\lambda x}\left(a_n \lambda^n + a_{n-1}\lambda^{n-1} \ldots + a_0\right) = 0$$

Thus $e^{\lambda x}$ is a solution if (since the exponential is never zero)

$$P(\lambda) = a_n \lambda^n + a_{n-1}\lambda^{n-1} \ldots + a_0 = 0$$

i.e. λ is a root of $P(\lambda)$, which is called the characteristic polynomial. By the fundamental theorem of algebra, we know that we'll always find n roots over the complex numbers. Thus we've found n solutions to the ODE

Why Is It Called The Characteristic Equation?   Recall from last time, we saw

$$ay'' + by' + cy = 0 \iff \dot{x} = \underbrace{\frac{1}{a}\begin{pmatrix} 0 & a \\ -c & -b \end{pmatrix}}_{=A} x \quad \text{where} \quad x = \begin{pmatrix} y \\ y' \end{pmatrix}$$

Notice that the characteristic equation, the one that determines the eigenvalues, is the same as the previous polynomial,

$$P(\lambda) = \det(A - 1\lambda) = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0 \implies a\lambda^2 + b\lambda + c = 0$$

Repeated Roots   On can check that you have repeated roots from your characteristic equation. To form a full basis for your solution space, i.e. the fundamental solutions, you can just stick t in front of the exponential for every solution you're missing. i.e. if $P(\lambda) = (\lambda - a)^3$, we'd have

$$y_1 = e^{at} \quad \& \quad y_2 = te^{at} \quad \& \quad y_3 = t^2 e^{at}$$

pg.220 - #17   Solve

$$y'' - 2ay' + a^2 y = 0$$

Solution   The characteristic polynomial for the ODE is

$$P(\lambda) = \lambda^2 - 2a\lambda + a^2 = (\lambda - a)^2$$

i.e. we have a repeated root of $\lambda = a$. Let's try $y(t) = te^{at}$ as a solution,

$$2a \underbrace{(e^{at} + a^2 te^{at})}_{y''} - 2a \underbrace{(e^{at} + ate^{at})}_{y'} + a^2 \underbrace{(te^{at})}_{y} = 0$$

Thus the general solution is given by

$$y(t) = c_1 e^{at} + c_2 te^{at}$$

$\square$

Example   Solve the IVP

$$3y''' + 5y'' + y' - y = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = -1$$

Solution   The characteristic equation for the ODE is

$$P(\lambda) = 3\lambda^3 + 5\lambda^2 + \lambda - 1 = (\lambda + 1)^2 (3\lambda - 1) = 0 \implies \lambda = \frac{1}{3} \quad \& \quad \lambda = -1(\text{ Repeated})$$

Thus general solution is given by

$$\boxed{y(t) = c_1 e^{-t} + c_2 te^{-t} + c_3 e^{t/3}}$$

The initial data implies

$$\begin{cases} c_1 + c_3 = 0 \\ -c_1 + c_2 + c_3/3 = 1 \\ c_1 + 2c_2 + c_3/9 = -1 \end{cases} \implies c_1 = -\frac{9}{16}, c_2 = \frac{1}{4}, c_3 = \frac{9}{16}$$

Hence the solution to the IVP is

$$\boxed{y(t) = \frac{9}{16} e^{t/3} + \left(\frac{t}{4} - \frac{9}{16} \right) e^{-t}}$$

$\square$

Complex Eigenvalues   In the case of complex eigenvalues, it seems we have a complex valued solution...though using Euler's Identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we may rewrite the solution in terms of real valued functions. Let $\lambda_+ = a + bi$ and $\lambda_- = a - bi$, then

$$\begin{aligned} y(t) = Ce^{\lambda_+ t} + \bar{C}e^{\lambda_- t} &= e^{at}(Ce^{ibt} + \bar{C}e^{-ibt}) \\ &= e^{at}((C + \bar{C})\cos(bt) + i(C - \bar{C})\sin(bt)) \\ &= e^{at}(c_1 \cos(bt) + c_2 \sin(bt)) \end{aligned}$$

Euler Equations   Consider

$$ax^2 y'' + bxy' + cy = 0 \quad x > 0$$

At first glance it seems like a foreign equation, but let's apply the change of variables $x \to z = \ln(x)$ to the ODE. We see via chain rule that

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \implies y' = \frac{1}{x}\frac{dy}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2 y}{dz^2}$$

Thus, the ODE in z becomes

$$a\frac{d^2 y}{dz^2} + (b - a)\frac{dy}{dz} + cy = 0$$

i.e. an ODE with constant coefficients. We know the solutions take exponential form...or in terms of the variable x we have

$$e^{\lambda z} = e^{\lambda \ln(x)} = e^{\ln(x^\lambda)} = x^\lambda$$

Thus we see trying $y(x) = x^\lambda$ will amount to the same ole story.