Tutorial Problems #2

MAT 267 – Advanced Ordinary Differential Equations – Fall 2014

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Solutions

pg.90 - # 7 Solve

$$(x^4y^2 - y)dx + (x^2y^4 - x)dy = 0$$

Solution Notice the symmetry, so lets check if the equation is exact. Let $M = x^4y^2 - y$ and $N = x^2y^4 - x$, then

$$M_y = 2x^4y - 1 \quad \& \quad N_x = 2xy^4 - 1$$

i.e. it's not exact, but we see

$$N_x - M_y = 2xy(y^3 - x^3) \quad \& \quad xM - yN = -x^2y^2(y^3 - x^3)$$

In a previous exercise we saw that

$$\mu(xy) = \exp\left(\int \frac{N_x - M_y}{xM - yN} d(xy)\right) = \exp\left(-2\int \frac{d(xy)}{xy}\right) = \exp -2\ln|xy| = \frac{1}{x^2y^2}$$

works as an integrating factor provide the function $N_x - M_y/xM - yN$ depended on xy, which in our case does! Thus the ODE becomes

$$\underbrace{\left(x^2 - \frac{1}{x^2y}\right)}_{=\tilde{M}} dx + \underbrace{\left(y^2 - \frac{1}{xy^2}\right)}_{=\tilde{N}} dy = 0$$

after multiplying by our integrating factor. It's easily seen that the ODE is now exact, so we integrate the components as usual.

$$\begin{aligned} F(x,y) &= \int \tilde{M} dx \oplus \int \tilde{N} dy \\ &= \int \left(x^2 - \frac{1}{x^2y}\right) dx \oplus \int \left(y^2 - \frac{1}{xy^2}\right) dy \\ &= \frac{x^3}{3} + \frac{1}{xy} \oplus \frac{y^3}{3} + \frac{1}{xy} \\ &= \frac{x^3 + y^3}{3} + \frac{1}{xy} \end{aligned}$$

Thus the general solution is

$$\boxed{\frac{x^3 + y^3}{3} + \frac{1}{xy} = C}$$

pg.90 - # 9 Solve

$$\underbrace{\left(\arctan(xy) + \frac{xy - 2xy^2}{1 + x^2y^2}\right)}_{M} dx + \underbrace{\frac{x^2 - 2x^2y}{1 + x^2y^2}}_{N} dy = 0$$

Solution We check if the equation is exact.

$$M_y = \frac{2x - 4xy}{1 + x^2y^2} - \frac{2x^3y^2 - 4x^3y^3}{(1 + x^2y^2)^2} = N_x$$

Since the equation is exact, we may integrate the components and take the linearity independent parts.

$$\begin{aligned} F(x,y) &= \int M dx \oplus \int N dy \\ &= x\arctan(xy) - \log(x^2y^2 + 1) \oplus x\arctan(xy) - \log(x^2y^2 + 1) \\ &= x\arctan(xy) - \log(x^2y^2 + 1) \end{aligned}$$

Thus the general solution is

$$\boxed{x\arctan(xy) - \log(x^2y^2 + 1) = C}$$

pg.103 - # 5 Solve

$$y' \sin y + \sin x \cos y = \sin x$$

Solution Notice if $z = \cos y$, then $z' = -y' \sin y$. Thus we're able to rewrite the ODE as

$$z' \underbrace{- \sin x}_{=p} z = \underbrace{- \sin x}_{=g}$$

In this form the ODE is first order linear. We know the solution is given by

$$z(x) = \frac{1}{\mu(x)} \int g(x)\mu(x)dx \quad \text{where} \quad \mu(x) = \exp\left(\int p(x)dx\right) = \exp\left(-\int \sin x dx\right) = \exp(\cos x)$$

Thus

$$z(x) = e^{-\cos x} \int -\sin x e^{\cos x} dx = e^{-\cos x}(e^{\cos x} + C) = 1 + Ce^{-\cos x}$$

In terms of the original function, we have

$$\cos(y) = 1 + Ce^{-\cos x} \implies \boxed{y(x) = \arccos(1 + Ce^{-\cos x})}$$

□

Picard Iterations Suppose you have a first order IVP. Using the fundamental theorem of calculus we see

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases} \iff y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$$

i.e the solution to the ODE is a solution to the integral equation. If we consider the RHS as an operator on our solution

$$T[g] = y_0 + \int_{t_0}^{t} f(s, g(s))ds$$

then existence of a solution to the ODE is equivalent to find a fixed point under this operator. i.e. $T[y] = y$. To show there exists some fixed point, lets try to define an approximating sequence that approaches such a point. Define the sequence as (Picard iterations)

$$\phi_0 = y_0 \quad \& \quad \phi_{k+1} = y_0 + \int_{t_0}^{t} f(s, \phi_k(s))ds$$

Its easy to show this limit converges if f is continuous (limits check out) and f is Lipschitz (allows us to bring the limit in the integral, i.e. $\lim \int = \int \lim$). Furthermore, one may show that T is a contraction map which allows us to apply the Banach Fixed point theorem to conclude the existence and uniqueness of y.

pg.726 - # 4 Find the first 3 Picard iterations of

$$\begin{cases} y' = 1 + xy \\ y(1) = 2 \end{cases}$$

Solution From the above, we see that $t_0 = 1$ and $y_0 = 2$, so $\phi_0 = 2$..

$$\phi_1 = 2 + \int_1^t f(s, \phi_0)ds = 2 + \int_1^t (1 + 2s)ds = t^2 + t$$

$$\phi_2 = 2 + \int_1^t f(s, \phi_1(s))ds = 2 + \int_1^t (1 + s(s^2 + s))ds = \frac{t^4}{4} + \frac{t^3}{3} + t + \frac{5}{12}$$

$$\phi_3 = 2 + \int_1^t f(s, \phi_2(s))ds = 2 + \int_1^t \left(1 + s\left(\frac{s^4}{4} + \frac{s^3}{3} + s + \frac{5}{12}\right)\right) ds = \frac{t^6}{24} + \frac{t^5}{15} + \frac{t^3}{3} + \frac{5t^2}{24} + t + \frac{7}{20}$$

These are the first 3 Picard iterations. Notice that in practice they're almost like building up a series expansion of the solution. □