Tutorial Problems #10

MAT 267 – Advanced Ordinary Differential Equations – Fall 2014

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Solutions

Taylor Series Method Example Solve using a series solution to 4th order:

$$\begin{cases} x^2y'' - 2xy' + \log(x)y = 0 \\ y(1) = 0 \\ y'(1) = 1/2 \end{cases}$$

Solution Recall that Taylor's Theorem states that $y \in C^\infty(B_\epsilon(x_0))$ may be represented around some x_0 as

$$y(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

The initial data gives us the first two terms in the series, the third is easily found from the ODE:

$$y'' = \frac{2}{x}y' - \frac{\log(x)}{x}y \implies y''(1) = 1$$

Differentiating the ODE will give the 3rd order term:

$$y^{(3)} = \frac{2}{x}y'' - \frac{2}{x^2}y' - \frac{1}{x^2}y + \frac{\log(x)}{x^2}y - \frac{\log(x)}{x}y' \implies y^{(3)}(1) = 1$$

If you differentiate once again, and check the value at $x_0 = 1$, we'll see

$$y^{(4)}(1) = -1$$

Plugging these values into the series expansion gives

$$\boxed{y(x) = \frac{(x-1)}{2} + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{24} + \mathcal{O}((x-1)^5)}$$

which is the series solution to 4th order.

Recurrence Equation Method Example Solve using a series solution to 5th order:

$$\begin{cases} y'' - xy' - y = \sin x \\ y(0) = a_0 \\ y'(0) = a_1 \end{cases}$$

Solution Suppose that

$$y(x) = \sum_{n=1}^{\infty} a_n x^n$$

solves the ODE. To find what the coefficients are, we substitute $y(x)$ into the ODE. First we solve the Homogenous part to the solution, i.e. $y'' - xy' - y = 0$. We see

$$\begin{aligned} y'' - xy' - y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= 2a_2 - a_0 + \sum_{n=1}^{\infty} \Big((n+2)(n+1)a_{n+2} - (n+1)a_n\Big)x^n \\ &= 0 \end{aligned}$$

By linear independence of our x^n's, we see the equation must have all of it's coefficients identically zero. i.e.

$$\boxed{a_{n+2} = \frac{a_n}{n+2} \quad \forall n \in \mathbb{N}}$$

This is called our recurrence equation for our series solution. One may easily check that we have

$$y(x) = a_0\left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \mathcal{O}(x^6)\right) + a_1\left(x + \frac{x^3}{3} + \frac{x^5}{15} + \mathcal{O}(x^7)\right)$$

is the homogenous solution to 5th order. To solve the non-homogenous solution, recall that sine has the following expansion

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Thus, reusing our previous computation, we want coefficients such that:

$$2a_2 - a_0 + \sum_{n=1}^{\infty} \Big((n+2)(n+1)a_{n+2} - (n+1)a_n\Big)x^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Since the right hand side only has odd coefficients, this forces all the even coefficients to die i.e. $a_{2n} = 0$ for all $n \in \mathbb{N}$. Thus the above becomes

$$\sum_{n=0}^{\infty} \Big((2n+3)(2n+2)a_{2n+3} - (2n+2)a_{2n+1}\Big)x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

i.e.

$$\boxed{(2n+3)(2n+2)a_{2n+3} - (2n+2)a_{2n+1} = \frac{(-1)^n}{(2n+1)!} \quad \forall n \in \mathbb{N}}$$

As an explicit example here, take $n = 0$, we see

$$6a_3 - 2a_1 = 1 \implies a_3 = \frac{1}{6} + \frac{a_1}{3}$$

For $n = 1$ we have

$$20a_5 - 4a_3 = -\frac{1}{6} \implies a_5 = -\frac{1}{40} + \frac{a_1}{15}$$

These terms with a_1 can be tossed into the homogenous solution, giving us that

$$\boxed{y(x) = a_0\left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \mathcal{O}(x^6)\right) + a_1\left(x + \frac{x^3}{3} + \frac{x^5}{15} + \mathcal{O}(x^7)\right) + \frac{x^3}{6} - \frac{x^5}{40} + \mathcal{O}(x^7)}$$

is the general solution to the ODE up to 5th order.

First Order ODE, Series Methods Solve using a series solution to 4th order:

$$\begin{cases} y' = \sin(xy) + x^2 \\ y(0) = 3 \end{cases}$$

Solution You could use Taylor Series and differentiate the ODE here, but lets try to brute force with series. Note that

$$\sin(x) = x - \frac{x^3}{6} + \mathcal{O}(x^5)$$

If we suppose that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

plugging this into the ODE gives

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = x\sum_{n=0}^{\infty} x^n - \frac{1}{6}\left(x\sum_{n=0}^{\infty} a_n x^n\right)^3 + x^2 + \mathcal{O}(x^5)$$

A quick expansion of the cubed series shows we have

$$\left(x\sum_{n=0}^{\infty} a_n x^n\right)^3 = a_0^3x^3 + 2(a_0^2a_1 + a_0a_1^2)x^4 + \mathcal{O}(x^5)$$

If we read things off term by term now, we obtain:

$$\begin{aligned} a_1 &= 0 && (1) \\ 2a_2 &= a_0 && (x) \\ 3a_3 &= (a_1 + 1) && (x^2) \\ 4a_4 &= a_2 - \frac{a_0^3}{6} && (x^3) \\ 5a_5 &= a_3 - \frac{a_0^2a_1 + a_0a_1^2}{3} && (x^4) \end{aligned}$$

Solving these equations yield

$$a_0 = 3, \quad a_1 = 0, \quad a_2 = \frac{3}{2}, \quad a_3 = \frac{1}{3}, \quad a_4 = -\frac{3}{4}$$

Thus our solution to 4th order is

$$\boxed{y(x) = 3 + \frac{3x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \mathcal{O}(x^5)}$$

First Order Systems, Series Methods Solve using a series solution to 4th order:

$$\begin{cases} x' = e^t + y, \quad y' = e^{-t} + x \\ x(0) = 0, \quad y(0) = 0 \end{cases}$$

Solution Suppose that

$$y(t) = \sum_{n=0}^{\infty} a_n t^n \quad \& \quad x(t) = \sum_{n=1}^{\infty} b_n t^n$$

and note

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

If we substitute the series into the system, we obtain the following from the first component:

$$b_1 + 2b_2 t + 3b_3 t^2 + 4b_4 t^3 + \mathcal{O}(t^4) = \underbrace{1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \ldots}_{e^t} + \underbrace{a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \mathcal{O}(t^4)}_{y}$$

This gives us the following relations

$$\begin{aligned} b_1 &= 1 + a_0 && (1) \\ 2b_2 &= 1 + a_1 && (t) \\ 3b_3 &= \frac{1}{2} + a_2 && (t^2) \\ 4b_4 &= \frac{1}{6} + a_3 && (t^3) \end{aligned}$$

Now the second component:

$$a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \mathcal{O}(t^4) = \underbrace{1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \ldots}_{e^{-t}} + \underbrace{b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \mathcal{O}(t^4)}_{x}$$

This gives:

$$\begin{aligned} a_1 &= 1 + b_0 && (1) \\ 2a_2 &= -1 + b_1 && (t) \\ 3a_3 &= \frac{1}{2} + b_2 && (t^2) \\ 4a_4 &= -\frac{1}{6} + b_3 && (t^3) \end{aligned}$$

Since we're given $a_0 = 0$ and $b_0 = 0$, it's easy to find that

$$a_1 = 1, \quad a_2 = 0, \quad a_3 = \frac{2}{9}, \quad a_4 = 0$$

$$b_1 = 1, \quad b_2 = 1, \quad b_3 = \frac{1}{6}, \quad b_4 = \frac{1}{6}$$

Plugging these into their respective series will give the system solution to 4th order. i.e.

$$\boxed{y(t) = x + \frac{2x^3}{9} + \mathcal{O}(x^5) \quad \& \quad x(t) = x + x^2 + \frac{x^3}{6} + \frac{x^4}{6} + \mathcal{O}(x^5)}$$

Quiz Question Solve using a series solution to 4th order:

$$\begin{cases} y' = y^2 - xy \\ y(0) = 2 \end{cases}$$

Solution This is set up very nicely for using Taylor Series. We know

$$y(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} x^n$$

is a solution. To find the coefficients, we use the ODE. We have

$$y' = y^2 - xy \implies y'(0) = 4$$

Differentiating the ODE gives

$$y'' = 2yy' - y - xy' \implies y''(0) = 2 * 2 * 4 - 2 = 14$$

We rinse and repeat

$$y^{(3)} = 2yy'' + 2(y')^2 - 2y' - xy'' \implies y^{(3)}(0) = 2 * 2 * 14 + 2 * (4)^2 - 2 * 4 = 80$$

One more time!

$$y^{(4)} = 2yy''' + 6y'y'' - 3y'' - xy''' \implies y^{(4)}(0) = 2 * 2 * 80 + 6 * 4 * 14 - 3 * 14 = 614$$

Thus our solution to 4th order is

$$\boxed{y(x) = 2 + 4x + 7x^2 + \frac{40x^3}{3} + \frac{307x^4}{12} + \mathcal{O}(x^5)}$$