MAT244H1 - Introduction to Ordinary Differential Equations
Summer 2016

Term Test - June 24, 2016, 2-4pm

Time allotted: 120 minutes. Aids permitted: None.

Full Name: ______________________________
Family Given

Student ID: ______________________________

Signature: ______________________________

Indicate which Tutorial Section you attend by filling in the appropriate circle:

◯ Tut 01 M 2-3 pm BA1240 Christopher Adkins

◯ Tut 02 M 3-4 pm BA1240 Fang Shalev Housfater

◯ Tut 05 W 5-6 pm BA2165 Krishan Rajaratnam

Instructions

- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 15 pages (including this title page). Make sure you have all of them.
- You can use pages 9-10 or the back of any page for rough work or extra space. If you continue a question on a different page, clearly indicate this.

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Bonus	Total
/20	/30	/20	/25	/20	/20	/15	/135

GOOD LUCK!

1)a) *[10 marks]* Solve this differential equation for $y(x)$:

$$-y' + xy = xy^2, \quad y(x=0) = \frac{1}{\pi + 1}$$

b) *[10 marks]* Now solve this differential equation for $y(x)$:

$$y' + xy = x \quad y(x=0) = \pi + 1$$

Solution:

a) We rewrite the equation as

$$\begin{aligned} -y' &= x\left(y^2 - y\right) \\ \frac{y'}{y - y^2} &= x \end{aligned}$$

and integrate both sides with respect to x. On the RHS:

$$\int x dx = \frac{x^2}{2} + c$$

On the LHS, we use partial fractions to write

$$\begin{aligned} \frac{1}{y(1-y)} &= \frac{A}{y} + \frac{B}{1-y} \\ &= \frac{A + (B - A)y}{y(1-y)} \end{aligned}$$

Thus $A = 1$ and $B = 1$. Changing the variable of integration from x to $y(x)$, $dy = y'(x)$, we have on the LHS

$$\begin{aligned} \int \frac{y'}{y - y^2} &= \int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy \\ &= \ln|y| - \ln|1-y| \\ &= \ln\left|\frac{y}{1-y}\right| \end{aligned}$$

Near our initial condition, $y(0) = \frac{1}{\pi+1}$, $y > 0$ and $1 - y > 0$, so we can remove the absolute values and put

$$\begin{aligned} \ln\left(\frac{y}{1-y}\right) &= \frac{x^2}{2} + c \\ \frac{y}{1-y} &= ke^{\frac{x^2}{2}} \end{aligned}$$

with $k = e^c$ set by initial conditions. Rearranging to solve for $y(x)$:

$$
\begin{aligned}
y(x) &= ke^{\frac{x^2}{2}} - y(x)ke^{\frac{x^2}{2}} \\
y(x)\left(1 + ke^{\frac{x^2}{2}}\right) &= ke^{\frac{x^2}{2}} \\
y(x) &= \frac{ke^{\frac{x^2}{2}}}{1 + ke^{\frac{x^2}{2}}}
\end{aligned}
$$

Plugging the initial condition,

$$
\begin{aligned}
\frac{1}{1+\pi} &= \frac{k}{1+k} \\
k &= \frac{1}{\pi}
\end{aligned}
$$

b) Here we use an integrating factor $\mu = e^{\int x dx} = e^{\frac{x^2}{2}}$. Thus

$$
\begin{aligned}
\left(ye^{\frac{x^2}{2}}\right)' &= xe^{\frac{x^2}{2}} \\
y(x)e^{\frac{x^2}{2}} &= \int xe^{\frac{x^2}{2}} dx \\
y(x) &= 1 + ce^{-\frac{x^2}{2}}
\end{aligned}
$$

where c is set by initial conditions. So

$$
\begin{aligned}
y(0) &= 1 + c \\
&= \pi + 1
\end{aligned}
$$

and $c = \pi$. Finally,

$$
y(x) = 1 + \pi e^{-\frac{x^2}{2}}
$$

2) a) *[10 marks]* Find the general solution to

$$y'' + 2y' + y = 0$$

b) *[20 marks]* Find the general solution to

$$y'' + 2y' + y = x + e^{-x}$$

Solution: a) Guessing the solution of the form $y = e^{rx}$, we get the characteristic equation:

$$r^2 + 2r + 1 = 0$$
$$(r+1)^2$$

So $r = -1$ is a double root and the independent solutions are $y_1 = e^{-x}$ and $y_2 = xe^{-x}$. The general solutions is

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

with c_1, c_2 set by initial conditions.

b) We just need to find one particular solution Y_p to solve

$$Y_p'' + 2Y_p' + Y_p = x + e^{-x}$$

As the right hand side has 2 parts we split $Y_p = Y_{p,1} + Y_{p,2}$ where $Y_{p,1}$ solves

$$Y_{p,1}'' + 2Y_{p,1}' + Y_{p,1} = x$$

and $Y_{p,2}$ solves

$$Y_{p,2}'' + 2Y_{p,2}' + Y_{p,2} = e^{-x}.$$

Finding $Y_{p,1}$ is easy. We guess $Y_{p,1} = Ax + B$ and solve for A, B. We have

$$Y_{p,1} = Ax + B$$
$$Y_{p,1}' = A$$
$$Y_{p,1}'' = 0$$

Thus

$$Y_{p,1}'' + 2Y_{p,1}' + Y_{p,1} = 2A + Ax + B$$
$$= x$$

Therefore $A = 1$ and $2A + B = 0$ so that $B = -2$. We conclude that

$$Y_{p,1} = x - 2$$

For $Y_{p,2}$ we can guess neither Ae^{-x} nor Axe^{-x} both of these functions appearing in the homogeneous solutions. We thus guess $Y_{p,2} = Ax^2e^{-x}$ and compute

$$\begin{aligned} Y_{p,2} &= Ax^2e^{-x} \\ Y'_{p,2} &= A\left(2x - x^2\right)e^{-x} \\ Y''_{p,2} &= A\left(2 - 4x + x^2\right)e^{-x} \end{aligned}$$

Plugging in the equation:

$$\begin{aligned} Y''_{p,2} + 2Y'_{p,2} + y_{p,2} &= \left(\left(A - 2A + A\right)x^2 + \left(-4A + 4A\right)x + 2A\right)e^{-x} \\ &= e^{-x} \end{aligned}$$

So $A = \frac{1}{2}$. Combining $Y_{p,1}$ and $Y_{p,2}$, we have a particular solution to the whole equation:

$$Y_p = x - 2 + \frac{1}{2}x^2e^{-x}$$

The general solution is obtained by adding a homogeneous part:

$$y(x) = c_1e^{-x} + c_2xe^{-x} + x - 2 + \frac{1}{2}x^2e^{-x}$$

where c_1 and c_2 are also set by initial conditions.

3) Suppose that the growth of population N is modeled by

$$\frac{dN}{dt} = N^3 - 3N^2 + 2N$$

a) *[10 marks]* Find and classify all equilibrium points (stable? unstable?)

b) *[10 marks]* Sketch graphs of solutions passing through the points $N(t=0) = \frac{1}{2}$, $N(t=25) = \frac{3}{2}$ and $N(t=100) = 4$.

Solutions:

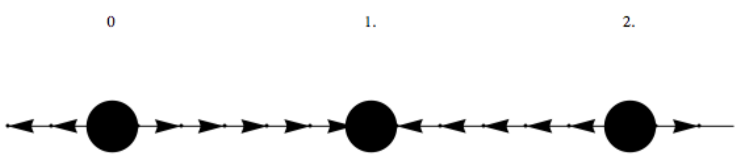
a) Letting $f(N) = N^3 - 3N^2 + 2N$ be the right hand side, the equilibrium points are the zeros of f. Factoring

$$\begin{aligned} f(N) &= N\left(N^2 - 3N + 2\right) \\ &= N(N-1)(N-2) \end{aligned}$$

Thus $N = 0$, $N = 1$ and $N = 2$ are the equilibrium points. To check their stability, we look at how the plot of $f(N)$ crosses the N axis at the given equilibrium point. One way to check this is to see what the derivative of f is doing. Since

$$f'(N) = 3N^2 - 6N + 2$$

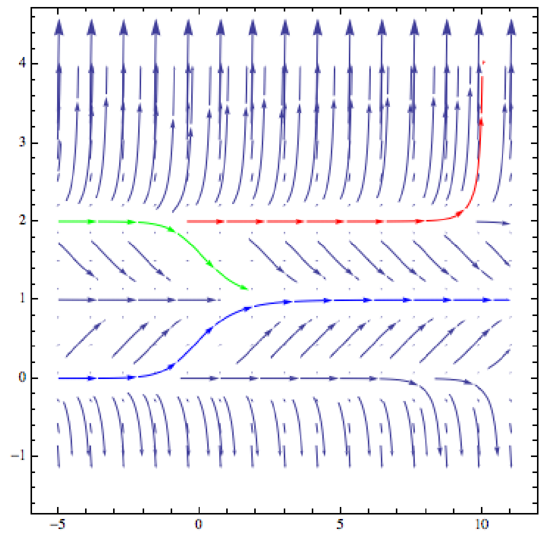
We have $f'(0) = 2 > 0$, $f'(1) = -1 < 0$ and $f'(2) = 2 > 0$. Thus $N = 0$ and $N = 2$ are unstable while $N = 1$ is a stable equilibrium point.



If $N(0) = \frac{1}{2}$ then $0 < N < 1$ and solutions tend to $N = 1$ asymptotically as it is the stable point ($N(t) \to 1$ as $t \to \infty$).

If $N(25) = \frac{3}{2}$, then $1 < N < 2$. As $t \to 0$, $N(t) \to N_0$ with $1 < N_0 < 2$. As $t \to \infty$, $N(t) \to 1$ as $N = 1$ is the stable equilibrium point.

If $N(100) = 4$, then as $t \to 0$, $N \to N_0 > 3$ (since solutions can't cross equilibrium lines by uniqueness). As $t \to \infty$, $N(t) \to \infty$ as well, since $f(N) = \frac{dN}{dt} > 0$ for any $N > 3$.



4) *[25 marks]* The average human has 20 litres of bodily fluid in them and will die if the level of alcohol in their bodily fluid exceeds 5 %. Suppose a sober person starts drinking 40% alcohol at the rate of 0.5 litres/hr. This person is also excreting bodily fluids at the same rate of 0.5 litres/hr. Assuming the alcohol mixes instantly with the bodily fluids, how long does it take this person to die?

We let $Q(t)$ denote the amount of alcohol in the person's body at t hours after they start drinking. Since they are sober initially: $Q(0) = 0$. At the time of death t_D, the amount is 5 % of 20 litres that is $Q(t_D) = 1$. Now we find the differential equation satisfied by Q to solve for t_D.

The rate of alcohol coming in is **constant** and is given by 40 % of 0.5 litres per hour that is

$$\text{rate in } = 0.4 \times 0.5 = 0.2$$

The rate of alcohol coming out is **variable** and depends on the amount already present: $\frac{Q(t)}{20}$. Therefore

$$\text{rate out } = \frac{Q(t)}{20} \times 0.5 = \frac{Q(t)}{40}$$

Finally we solve

$$\frac{dQ}{dt} = 0.2 - \frac{Q(t)}{40}, \quad Q(0) = 0$$

$$\frac{dQ}{dt} + \frac{1}{40}Q(t) = 0.2$$

This is a standard linear equation. We use the integrating factor $\mu = e^{t/40}$ to write

$$\left(Q(t)e^{t/40}\right)' = 0.2e^{t/40}$$

$$Q(t)e^{t/40} = \int 0.2e^{t/40}dt$$

$$Q(t)e^{t/40} = 40 \times 0.2e^{t/40} + c$$

$$Q(t) = 8 + ce^{-t/40}$$

Imposing the initial conditions to solve for c, we get

$$Q(0) = 0 = 8 + c$$

$$c = -8$$

Finally we find the time of death t_D, that is when $Q(t_D) = 1$:

$$Q(t_D) = 8\left(1 - e^{-t_D/40}\right) = 1$$

$$e^{-t_D/40} = \frac{7}{8}$$

$$t_D = -40\ln\left(\frac{7}{8}\right)$$

For curiosity, if you had a calculator you would find $t_D \approx 5.3$ hours.

5) *[20 marks]* Suppose $y_1(x)$ and $y_2(x)$ are linearly independent solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

where $p(x)$ and $q(x)$ are given continuous functions of x. Prove that $u_1 = y_1 - y_2$ and $u_2 = y_1 + y_2$ are also linearly independent solutions of (1)

Solution: Recall that the condition for linear independence of any two functions y_1 and y_2 is that the Wronskian $W \neq 0$ where

$$W = \det \begin{pmatrix} y_1 & y_2 \\ y1' & y_2' \end{pmatrix}$$

We check that u_1 and u_2 are linearly independent

$$\begin{aligned} W_{[u_1,u_2]} &= \det \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} \\ &= \det \begin{pmatrix} (y_1 - y_2) & (y_1 + y_2) \\ (y_1' - y_2') & (y_1' + y_2') \end{pmatrix} \\ &= (y_1 - y_2)\left(y_1' + y_2'\right) - (y_1 + y_2)\left(y_1' - y_2'\right) \\ &= y_1y_1' + y_1y_2' - y_2y_1' - y_2y_2' \\ &\quad - y_1y_1' + y_1y_2' - y_2y_1' + y_2y_2' \\ &= 2\left(y_1y_2' - y_2y_1'\right) \\ &= 2\det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2 \end{pmatrix} \\ &= 2W_{[y_1,y_2]} \end{aligned}$$

And by assumption, y_1 and y_2 are linearly independent and therefore $W_{[y_1,y_2]} \neq 0$. The same is therefore true for $W_{[u_1,u_2]}$ and so u_1, u_2 are linearly independent functions.

6) Consider the Legendre equation for $y(x)$:

$$\left(1-x^2\right)y''-2xy'+2y=0 \tag{2}$$

a) *[5 marks]* Show that the function $y(x)=x$ solves this equation

b) *[15 marks]* Find another independent solution to this equation.

Solution:

a) If $y(x)=x$, then $y'(x)=1$ and $y''(x)=0$ so that

$$\left(1-x^2\right)y''-2xy'+2y=-2x+2x=0$$

and $y(x)=x$ is a solution.

b) One way to look for an independent solution is to write $y=xv(x)$ and try solving for $v(x)$. This is called reduction of order. We then compute

$$\begin{aligned} y &= xv(x) \\ y' &= v(x)+xv'(x) \\ y'' &= 2v'(x)+xv''(x) \end{aligned}$$

Then

$$\begin{aligned} \left(1-x^2\right)y''-2xy'+2y &= \left(1-x^2\right)\left(2v'+xv''\right)-2x\left(v+xv'\right)+2xv \\ &= x\left(1-x^2\right)v''+2\left(1-x^2\right)v'-2x^2v'-2xv+2xv \\ &= x\left(1-x^2\right)v''+2\left(1-2x^2\right)v'=0 \end{aligned}$$

Note that all the terms with no derivative in v cancelled and we have reduced to a first order system. We can rewrite the last equation as

$$\begin{aligned} \frac{v''}{v'} &= \frac{-2\left(1-2x^2\right)}{x\left(1-x^2\right)} \\ &= \frac{-2\left(1-x^2\right)+2x^2}{x\left(1-x^2\right)} \\ &= \frac{-2}{x}+\frac{2x^2}{x\left(1-x^2\right)} \\ &= \frac{-2}{x}+\frac{2x}{1-x^2} \end{aligned}$$

Integrating both sides with respect to x: on the LHS set $v'=v'(x)$ so that $dv'=v''(x)dx$ and

$$\int\frac{v''}{v'}dx=\ln v'$$

On the right hand side we integrate:

$$\int \left(\frac{-2}{x} + \frac{2x}{1-x^2}\right) dx = -2\ln x - \ln\left(1-x^2\right)$$
$$= \ln\left(\frac{1}{x^2\left(1-x^2\right)}\right)$$

Therefore

$$v'(x) = \frac{1}{x^2\left(1-x^2\right)}$$

In problem 1)a), we have done already the partial fraction decomposition for the right hand side (just replace y by x^2) therefore

$$v'(x) = \frac{1}{x^2} + \frac{1}{1-x^2}$$
$$v(x) = \int \left(\frac{1}{x^2} + \frac{1}{1-x^2}\right) dx$$
$$v(x) = -\frac{1}{x} + \int \frac{dx}{1-x^2}$$

If you don't remember the final integral, my favorite method is trig substitution. Say $x = \sin\theta$, so that $1-x^2 = \cos^2\theta$ then $dx = \cos\theta d\theta$ and we have

$$\int \frac{dx}{1-x^2} = \int \frac{\cos\theta d\theta}{\cos^2\theta}$$
$$= \int \sec\theta$$
$$= \ln\left(\sec\theta + \tan\theta\right)$$

Since $x = \sin\theta$, we have $\sec\theta = \frac{1}{\sqrt{1-x^2}}$ and $\tan\theta = \frac{x}{\sqrt{1-x^2}}$. Thus

$$\int \frac{dx}{1-x^2} = \ln\left(\frac{1+x}{\sqrt{1-x^2}}\right)$$

To summarize, we have that

$$v(x) = -\frac{1}{x} + \ln\left(\frac{1+x}{\sqrt{1-x^2}}\right)$$

We recall our ansatz: to look for a solution in the form $y(x) = xv(x)$. Therefore

$$y(x) = -1 + x\ln\left(\frac{1+x}{\sqrt{1-x^2}}\right)$$

is another linearly independent solution

BONUS

a) *[5 marks]* Solve this differential equation for $y(x)$.

$$y' = (x+y)^2, \quad y(0) = 1$$

(Hint: it is neither separable nor homogenous)

Solution:

The idea is to make a change of variables that resembles the right hand side. Setting $v(x) = x + y(x)$, we compute

$$\begin{aligned} v'(x) &= y'(x) + 1 \\ &= v^2 + 1 \end{aligned}$$

This is now a separable equation so

$$\begin{aligned} \frac{v'}{v^2+1} &= 1 \\ \arctan v &= x + c \\ v(x) &= \tan(x+c) \end{aligned}$$

Solving for $y(x) = v(x) - x$ we have

$$y(x) = \tan(x+c) - x$$

where c is from the initial condition $y(0) = 1$:

$$y(0) = \tan c = 1 \implies c = \frac{\pi}{4} \tag{3}$$

b) *[10 marks]* Show that the solution $y(t)$ to

$$y'' + \omega_0^2 y = \cos(\omega t), \quad y'(0) = 0, \quad y(0) = 2A$$

can be written as

$$y = 2A\cos\left(\frac{\omega_0+\omega}{2}t\right)\cos\left(\frac{\omega_0-\omega}{2}t\right)$$

so long as $\omega_0 \neq \omega$. What is the value of A? Draw a sketch of this solution when $|\omega_0 - \omega|$ is really small. Explain what happens as $\omega \to \omega_0$.

Solution: If $\omega \neq \omega_0$, we use the method of undetermined coefficients- and look for a particular solution of the form $Y_p = A\cos(\omega t) + B\sin(\omega t)$. We compute then

$$Y_p = A\cos(\omega t) + B\sin(\omega t)$$
$$Y_p' = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$$
$$Y_p'' = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$$

Plugging into the equation , we have

$$\begin{aligned} Y_p'' + \omega_0^2 Y_p &= A\left(\omega_0^2 - \omega^2\right)\cos(\omega t) + B\left(\omega_0^2 - \omega^2\right)\sin(\omega t) \\ &= \cos(\omega t) \end{aligned}$$

Therefore $A = \frac{1}{\omega_0^2 - \omega^2}$ while $B = 0$. The general solution is now

$$y(t) = c_1\sin(\omega_0 t) + c_2\cos(\omega_0 t) + A\cos(\omega t)$$

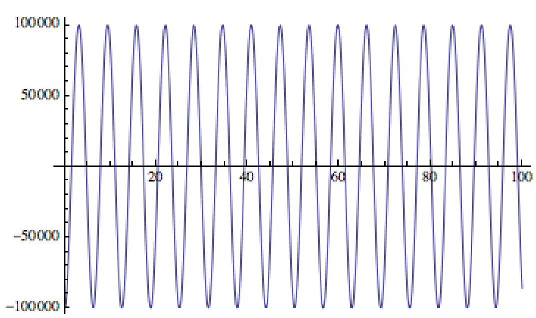
where c_1 and c_2, we impose by the boundary conditions:

$$y'(0) = \omega_0 c_1 = 0 \implies c_1 = 0$$
$$y(0) = c_2 + A = 2A \implies c_2 = A$$

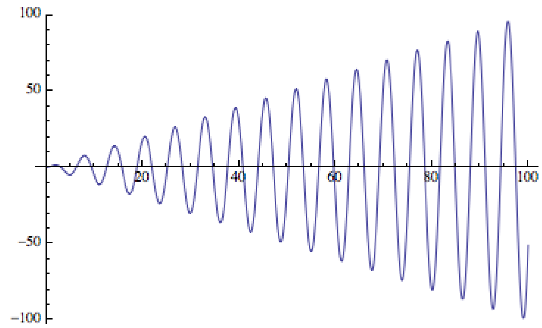
Therefore this solution reads

$$\begin{aligned} y(t) &= A\left(\cos(\omega_0 t) + \cos(\omega t)\right) \\ &= 2A\cos\left(\frac{\omega + \omega_0}{2}t\right)\cos\left(\frac{\omega_0 - \omega}{2}t\right) \end{aligned}$$

the last line being a standard trig identity and $A = \frac{1}{\omega_0^2 - \omega^2}$.



As $\omega \to \omega_0$, we can no longer guess $Y_p = A\cos(\omega t) + B\sin(\omega t)$. Instead, we find resonant solutions of the form $Y_p = Ct\sin(\omega_0 t)$ that grow without bound as $t \to \infty$.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