MAT244 - Ordinary Differential Equations - Summer 2016
Assignment 3 Due: August 3, 2016

Full Name: ______________________________________

Last First

Student #: ______________________________________

Indicate which Tutorial Section you attend by filling in the appropriate circle:

◯ Tut 01 M 2-3 pm BA1240 Christopher Adkins

◯ Tut 02 M 3-4 pm BA1240 Fang Shalev Housfater

◯ Tut 05 W 5-6 pm BA2165 Krishan Rajaratnam

Print out this page, fill it out and attach it to the front of your assignment.

Instructions:

- August 3, 2016 at the start of the lecture 13:10pm in BA1170.
- You may collaborate with your classmates but you MUST write up your solutions independently.
- Write your solutions clearly, showing all steps. Do not submit just your rough work. Grading is based on both the correctness and thee presentation of your answer.
- Late assignments will not be accepted without appropriate documentation to explain the lateness (eg. a UofT medical note)
- Assignments may be submitted to the course instructor for remarking up to one week after they are returned. If you request a regrade, please attach a note explaining clearly which part and why you believe it was graded incorrectly.

For grader use:

Q1	Q2	Q3	Q4	Q5	Total

each problem is worth 20 points for a total of 100.

1) Consider the nonhomogeneous problem

$$\mathbf{y}' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} t \\ 0 \end{pmatrix}$$

a) By solving first the homogeneous equation $\mathbf{y}' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{y}$, find a fundamental matrix for this equation. What is the phase portrait of the homogenous system? Is this stable?

Solution: Finding the fundamental matrix means to fin 2 independent solutions:

The eigenvalue equation for

$$\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

gives

$$\begin{aligned}
(-1-\lambda)^2 + 1 &= 0 \\
\lambda^2 + 2\lambda + 2 &= 0 \\
\lambda &= \frac{1}{2}\left(-2 \pm \sqrt{4-8}\right) \\
\lambda &= \frac{1}{2}\left(-2 \pm 2i\right) \\
\lambda &= \frac{1}{2}\left(-2 \pm 2i\right) \\
\lambda &= -1 \pm i
\end{aligned}$$

To find the fundamental matrix, we look for one of the eigenvectors:

$$\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = (-1+i) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\begin{cases} -\xi_1 + \xi_2 & = (-1+i)\,\xi_1 \\ -\xi_1 - \xi_2 & = (-1+i)\,\xi_2 \end{cases}$$

and both of the above equations reduce to

$$\xi_2 = i\xi_1$$

Thus one eigenvector of the system is $\xi_1 = 1$, $\xi_2 = i$ and the corresponding solution is

$$y^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1+i)t}$$

and the other solution is just a complex conjugate. Separating as usual, the real and imaginary parts, we have:

$$\begin{aligned} \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1+i)t} &= e^{-t} \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos t + i \sin t) \\ &= e^{-t} \left(\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \right) \end{aligned}$$

Thus the general solution can be written as

$$e^{-t} \left(c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \right)$$

and the fundamental matrix reduces to

$$\begin{pmatrix} e^{-t} \cos t & e^{-t} \sin t \\ -e^{-t} \sin t & e^{-t} \cos t \end{pmatrix}$$

b) Use the fundamental matrix and variation of parameters to find the general solution to the non homogenous system.

Using our formula for nonhomogeneous solutions, we first find the inverse of the fundamental matrix.

$$\Psi = \begin{pmatrix} e^{-t} \cos t & e^{-t} \sin t \\ -e^{-t} \sin t & e^{-t} \cos t \end{pmatrix}$$

The determinant of Ψ is easy to find:

$$\begin{aligned} \det \Psi &= e^{-2t} \left(\cos^2 t + \sin^2 t \right) \\ &= e^{-2t} \end{aligned}$$

thus

$$\begin{aligned} \Psi^{-1} &= e^{2t} \begin{pmatrix} e^{-t} \cos t & -e^{-t} \sin t \\ e^{-t} \sin t & e^{-t} \cos t \end{pmatrix} \\ &= \begin{pmatrix} e^{t} \cos t & -e^{t} \sin t \\ e^{t} \sin t & e^{t} \cos t \end{pmatrix} \end{aligned}$$

Usinv variation of parameters: we look for $\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ that satisfies the equation

$$\mathbf{y}^{(p)} = \Psi \int \Psi^{-1} g$$

We compute first the integrand:

$$\begin{aligned} \Psi^{-1} g &= \begin{pmatrix} e^t \cos t & -e^t \sin t \\ e^t \sin t & e^t \cos t \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} te^t \cos t \\ te^t \sin t \end{pmatrix} \end{aligned}$$

Integrating this function is somewhat difficult. But if we know the classic integrals (just integrate by parts twice):

$$\begin{aligned} \int e^t \cos t &= \frac{1}{2} e^t (\sin t + \cos t) \\ \int e^t \sin t &= \frac{1}{2} e^t (\sin t - \cos t) \end{aligned}$$

it is not so bad. To solve

$$\int te^t \cos t$$

we set $du = e^t \cos t$ and $v = t$ and integrate by parts:

$$\begin{aligned} \int te^t \cos t &= \frac{1}{2} te^t (\sin t + \cos t) - \frac{1}{2} \int e^t (\cos t + \sin t) \\ &= \frac{1}{2} te^t (\sin t + \cos t) - \frac{1}{4} e^t (\cos t - \sin t) - \frac{1}{4} e^t (\cos t + \sin t) \\ &= \frac{1}{2} te^t (\sin t + \cos t) - \frac{1}{2} e^t \sin t \end{aligned}$$

Similarly we compute

$$\begin{aligned} \int te^t \sin t &= \frac{1}{2} te^t (\sin t - \cos t) - \frac{1}{2} \int e^t ((\sin t - \cos t)) \\ &= \frac{1}{2} te^t (\sin t - \cos t) - \frac{1}{4} e^t (\sin t - \cos t) + \frac{1}{4} e^t (\sin t + \cos t) \\ &= \frac{1}{2} te^t (\sin t - \cos t) + \frac{1}{2} e^t \cos t \end{aligned}$$

The particular solution is found by multiplying by our fundamental matrix

$$\begin{aligned}\mathbf{y}^{(p)} &= \begin{pmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\sin t & e^{-t}\cos t \end{pmatrix} \begin{pmatrix} \frac{1}{2}te^{t}\left(\sin t + \cos t\right) - \frac{1}{2}e^{t}\sin t \\ \frac{1}{2}te^{t}\left(\sin t - \cos t\right) + \frac{1}{2}e^{t}\cos t \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} t \\ -t+1 \end{pmatrix}\end{aligned}$$

The general solution is written as

$$\mathbf{y} = \Psi \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \mathbf{y}^{(p)}$$

with c_1, c_2 set by initial conditions

Mark this one easy obviously

1)c) Draw an <u>approximate</u> phase portrait of the nonhomogenous system. Is this system stable? Explain why or why not!

Insert sketch

Solution: the main idea is that we can determine the behaviour as $t \to \pm\infty$. Indeed in that case $e^{-t} \to 0$ and the homogeneous part of the solution tends to zero asymptotically.

The terms in the particular solution then become important. In particular the term

$$\frac{1}{2}\begin{pmatrix} t \\ -t+1 \end{pmatrix}$$

increases in magnitude and dominates the solution for large t.

as $t \to -\infty$, the homogeneous solution dominate because of the $e^{-}t$ terms and we have the expected spiral behaviour.

(Any sketch recognizing this and that makes sense shold get full marks!)

2) Find the Jacobian matrix of the following vector valued functions of $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ (this is for those students that never learned multivariable calculus. If you have, welcome to free marks!):

a) $\mathbf{f}(\mathbf{y}) = \begin{pmatrix} y_1 e^{y_2} \\ y_2 e^{y_1} \end{pmatrix}$

Solution: The Jacobian of a vector function is given by

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}$$

Here f_1 is the first component of $\mathbf{f}(\mathbf{y})$ and f_2 the second component

For this question:

$$J = \begin{pmatrix} e^{y_2} & y_1 e^{y_2} \\ y_2 e^{y_1} & e^{y_1} \end{pmatrix}$$

b)$\mathbf{f}(\mathbf{y}) = \begin{pmatrix} \sin(y_1 y_2) \\ \cos(y_1 y_2) \end{pmatrix}$

Solution:

$$J = \begin{pmatrix} y_2 \cos(y_1 y_2) & y_1 \cos(y_1 y_2) \\ -y_2 \sin(y_1 y_2) & -y_1 \sin(y_1 y_2) \end{pmatrix}$$

c) $\mathbf{f}(\mathbf{y}) = \begin{pmatrix} \frac{y_1}{y_1^2+y_2^2} \\ \frac{y_2}{y_1^2+y_2^2} \end{pmatrix}$

Solution:

$$J = \begin{pmatrix} \frac{y_2^2-y_1^2}{\left(y_1^2+y_2^2\right)^2} & \frac{-2y_1y_2}{\left(y_1^2+y_2^2\right)^2} \\ \frac{-2y_1y_2}{\left(y_1^2+y_2^2\right)^2} & \frac{y_1^2-y_2^2}{\left(y_1^2+y_2^2\right)^2} \end{pmatrix}$$

3) In this question, we analyze the equation for an electric circuit (you may find the end of 3.7 mildly helpful. Elsewise consider your Griffiths "introduction to electrodynamics" text or any equivalent that is freely available). In a circuit the quantity of interest is the current $I(t)$. Assuming that we know the capacitance (C), inductance (L) and resistance (R) in a given circuit. We assume all three of these quantities are constant. The equation for $I(t)$ then simplifies to

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0 \tag{1}$$

3)a) The equation (1) is a linear second order equation with constant coefficients. Guessing a solution $I(t) = e^{\lambda t}$ find the characteristic equation satisfied by λ

Solution: I will assume that all the constants L, R and C are positive for simplicity plugging the usual ansatz $I(t) = e^{\lambda t}$ into the equation (1) gives

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0$$

which is the characteristic equation we saw in chapter 3. The solutions for λ using the quadratic test:

$$\lambda = \frac{1}{2L}\left(-R \pm \sqrt{R^2 - 4\frac{L}{C}}\right)$$

3)b) Now let us transform (1) into a vector system of equations. Set $y_1(t) = I(t)$ and $y_2(t) = \frac{dI}{dt}(t)$. Let $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Find the matrix A such that

$$\mathbf{y}' = A\mathbf{y} \tag{2}$$

Find the eigenvalues of A and relate them to the equation you found in part a).

Setting $y_1 = I(t)$ and $y_2 = I'(t)$, we see that $y_1'(t) = I'(t) = y_2(t)$.
Similarly, $y_2'(t) = I''(t) = -\frac{R}{L}I' - \frac{I}{LC} = -\frac{R}{L}y_2 - \frac{y_1}{LC}$ where we just used equation (1). Therefore

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix}$$

We find the eigendvalues of A in the usual way: setting $0 = \det(A - \lambda I)$:

$$\begin{aligned} \det\begin{pmatrix} -\lambda & 1 \\ -\frac{1}{LC} & -\frac{R}{L} - \lambda \end{pmatrix} &= -\lambda\left(-\frac{R}{L} - \lambda\right) + \frac{1}{LC} \\ &= \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0 \end{aligned}$$

Therefore by the quadratic formula

$$\begin{aligned} \lambda &= \frac{1}{2}\left(-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - 4\frac{1}{LC}}\right) \\ &= \frac{1}{2}\left(-\frac{R}{L} \pm \frac{1}{L}\sqrt{R^2 - 4\frac{L}{C}}\right) \end{aligned}$$

This is exactly the same equation we obtained in part a)!

3)c) Clearly the quantity $\left(R^2 - 4\frac{L}{C}\right)$ plays an important role. Draw a phase portrait of (2) when $R^2 > 4\frac{L}{C}$ and when $R^2 < 4\frac{L}{C}$. (Note: you aren't forced to solve (2) explicitly in either case)

By looking at the Eigenvalues, it is clear that we have a stable node when $R^2 > 4\frac{L}{C}$ and a stable spiral when $R^2 < 4\frac{L}{C}$.

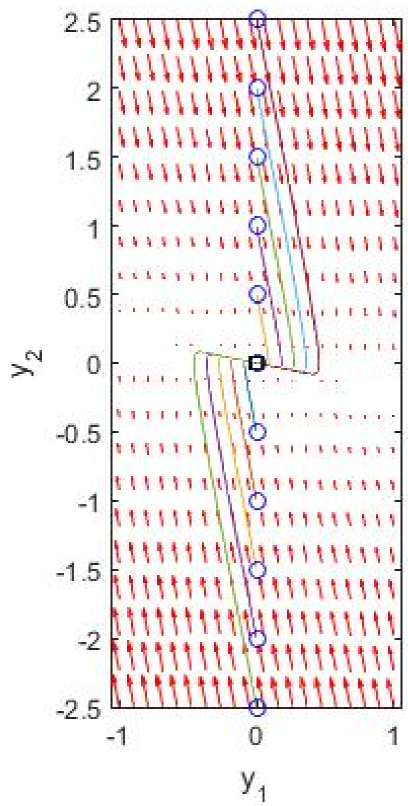

Figure 1: A typical phase plot for the stable node when $R^2 > 4\frac{L}{C}$

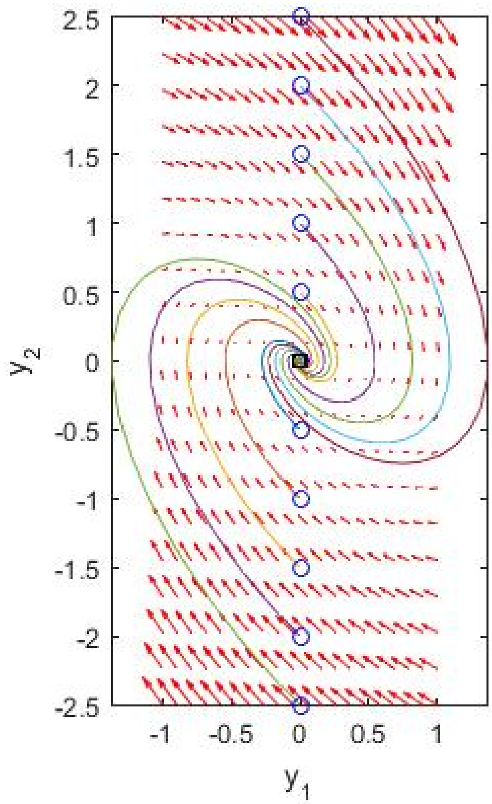

Figure 2: A typical phase plot for the stable spiral when $R^2 4\frac{L}{C}$

We plot an example for each case. In the first, let me take $L = C = 1$ and $R = 5$ and the result is in Figure 1:

In the second, let's take $R = L = C = 1$ and the result is in Figure 2

3)d) Use your phase portraits in part 3)c) to draw a sketch of $I(t)$ in the two cases: $R^2 > 4\frac{L}{C}$ and $R^2 < 4\frac{L}{C}$.

Solution: There is not much to do here. Just translate the x-coordinate. For example from Figure 3, we get the following for $I(t)$:

Similarly, from Figure 2, we get Figure 4

Note that the second figure shows oscillatory behaviour while the first one does not.

3)e) Now return to (1) and solve for the general solution $I(t)$. Is your solution consistent with your sketches in part 3)d)? Explain why or why not.

Solution: Using our methods in chapter 3, we know how to solve a second order ode. First, in the case that $R^2 > 4\frac{L}{C}$, we have from

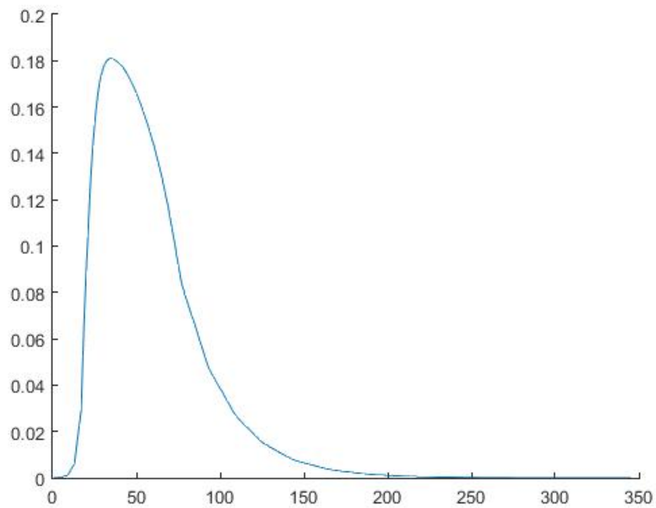

Figure 3: Plot of the y_1 coordinate corresponding to $I(t)$ in the Figure 1

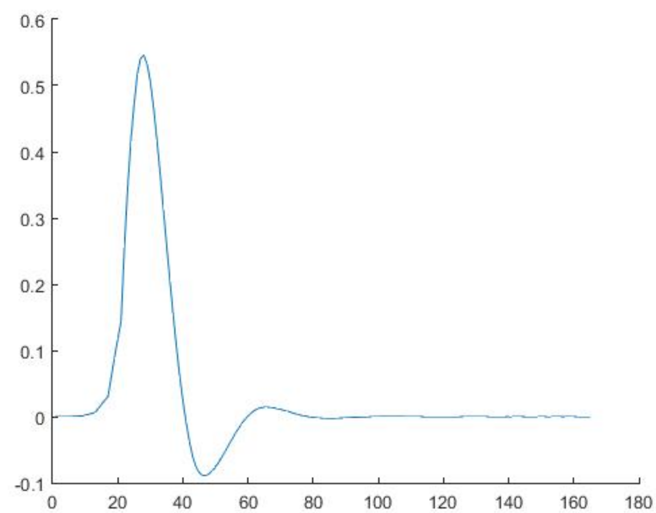

Figure 4: Plot of the y_1 coordinate corresponding to $I(t)$ in the Figure 2

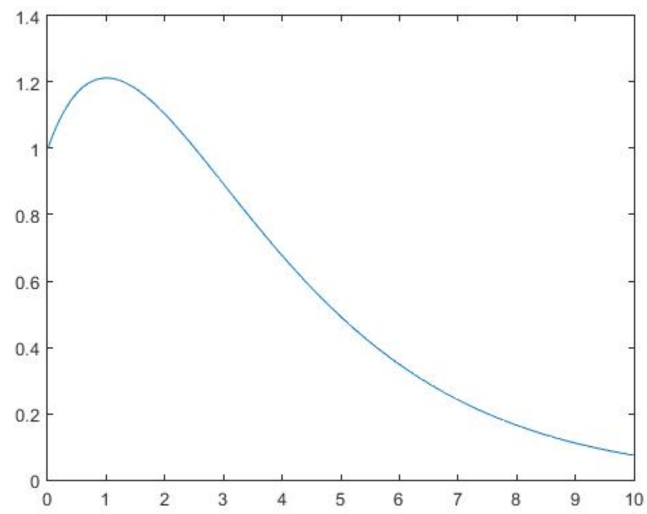

Figure 5: A typical solution in the critical case $R^2 = 4\frac{L}{C}$

$$\lambda = \frac{1}{2L}\left(-R \pm \sqrt{R^2 - 4\frac{L}{C}}\right)$$

the general solution

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Where $\lambda_1 = \frac{1}{2L}\left(-R + \sqrt{R^2 - 4\frac{L}{C}}\right)$ and $\lambda_2 = \frac{1}{2L}\left(-R - \sqrt{R^2 - 4\frac{L}{C}}\right)$. And since both of the eigenvalues are negative, the solution tends to zeros as $t \to \infty$. You can try plotting solutions for various c_1, c_2 and get a similar picture to 3.

Similarly, in the case that $R^2 < 4\frac{L}{C}$, the eigenvalues are complex conjugates and we have

$$y(t) = e^{-\frac{R}{2L}}\left(c_1 \sin\left(\sqrt{4\frac{L}{C} - R^2}\right) + c_2 \cos\left(\sqrt{4\frac{L}{C} - R^2}\right)\right)$$

3)f) Explain what happens if $R^2 = 4\frac{L}{C}$. What does the solution look like in this critical case?

In the critical case, we look for another solution in the form $te^{\lambda t}$. We have then a solution of the form:

$$y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

where $\lambda = \frac{-R}{2L}$ is the only eigenvalue we have. Note that for any c_1, c_2 the solution tends to zero but the second term tends to zero slower.

You can experiment with different values of c_1 and c_2 and obtain a graph similar to Figure 5

4) Consider the nonlinear system of equations for $x(t)$, $y(t)$:

$$\begin{cases} \frac{dx}{dt} & = x - y + \sin\frac{\pi x}{2} \\ \frac{dy}{dt} & = \cos\frac{\pi x}{2} \end{cases}$$

a) Find all the equilibrium points of this system

Equilibrium points are when both $x'(t)$ and $y'(t)$ are zero. It is easiest to solve for points where $x'(t) = 0$, that is when

$$\sin\left(\frac{\pi x}{2}\right) = 0$$

So we can take any odd number for x: $x = 1, 3, 5, ..., 2n+1, ..$ For $y'(t) = 0$ is a bit more complicated: we solve

$$2n + 1 - y + \sin\left(\frac{\pi\,(2n+1)}{2}\right)$$

Note that $\sin\left(\frac{\pi(2n+1)}{2}\right) = (-1)^n$ If $n = 0$, we have that $y = 2$ if $n = 1$, then again we have $x = 2$. In general when $n = 2k$, $y = 4k + 2$ and if $n = 2k + 1$, then $y = 4k + 2$ as well. Our points of equilibrium thus follow the following pattern:

$$(1, 2), \quad (3, 2), \quad (5, 6), \quad (7, 6).....$$

b) Linearize the system near each equilibrium point (that means find the linear homogenous problem solved by $\tilde{x} = x - x_0$, $\tilde{y} = y - y_0$ near each equilibrium point (x_0, y_0)).

Solution: Here we have to compute the Jacobian near each equilibrium point. The lucky thing is that there are only 2 forms for the Jacobian. Let us compute it: in general we have

$$J = \begin{pmatrix} 1 + \frac{\pi}{2}\cos\frac{\pi x}{2} & -1 \\ -\frac{\pi}{2}\sin\frac{\pi x}{2} & 0 \end{pmatrix}$$

Evaluating at various values of x, we have either

$$J = \begin{pmatrix} 1 & -1 \\ -\frac{\pi}{2} & 0 \end{pmatrix}$$

for $x = 1, 5, 9, ..., 4k+1, ..$

Or we have

$$J = \begin{pmatrix} 1 & -1 \\ \frac{\pi}{2} & 0 \end{pmatrix}$$

for $x = 3, 7, 11, ..., 4k+3, ...$

c) Classify the type (saddlepoint, node, etc...) and stability of each equilibrium point

Solution: If $x = 4k+1$, we have

$$J = \begin{pmatrix} 1 & -1 \\ -\frac{\pi}{2} & 0 \end{pmatrix}$$

Finding the eigenvalues gives

$$\begin{aligned} \det \begin{pmatrix} 1-\lambda & -1 \\ -\frac{\pi}{2} & -\lambda \end{pmatrix} &= (1-\lambda)(-\lambda) - \frac{\pi}{2} \\ &= \lambda^2 - \lambda - \frac{pi}{2} \end{aligned}$$

In this case the quadratic formula gives

$$\lambda = \frac{1}{2}\left(1 \pm \sqrt{1+2\pi}\right)$$

Since $2\pi > 1$, therefore one eigenvalue (with the plus sign) is positive and the other (with the negative sign) is negative. Therefore this is a saddle point.

Similarly if $x = 4k+3$ then we find the eigenvalues of

$$J = \begin{pmatrix} 1 & -1 \\ \frac{\pi}{2} & 0 \end{pmatrix}$$

That is we solve

$$\begin{aligned} \det \begin{pmatrix} 1-\lambda & -1 \\ \frac{\pi}{2} & -\lambda \end{pmatrix} &= (1-\lambda)(-\lambda) + \frac{\pi}{2} \\ &= \lambda^2 - \lambda + \frac{\pi}{2} \end{aligned}$$

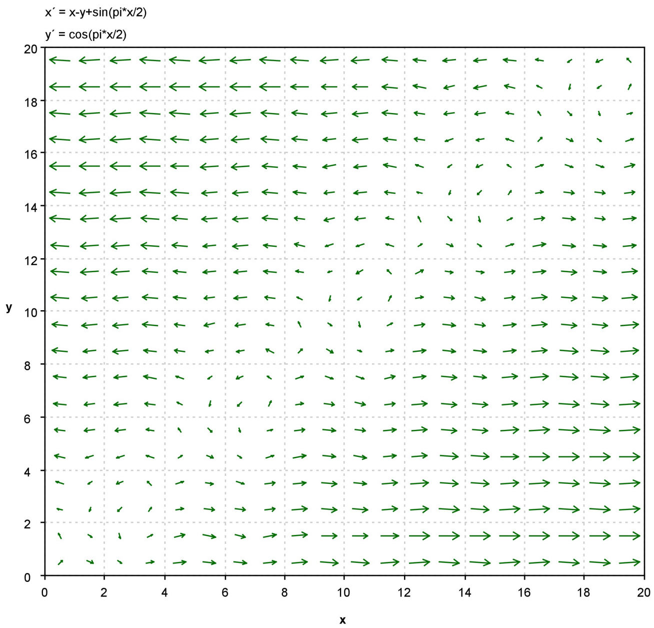

Figure 6: A direction field for the nonlinear system of problem 4

Therefore the eigenvalues in this case are complex conjugates:

$$\lambda = \frac{1}{2}\left(1 \pm i\sqrt{2\pi - 1}\right)$$

Since the real part is positive, this is an unstable spiral.

Since these are the only choices for the jacobian, there will be no other linear phase portraits near other equilibrium points.

d) Draw an approximate phase portrait of the full nonlinear system (include your linearized system near each point).

Solution: See Figure 6 for a direction field of this system. So long as your solution reasonably followed these directions you should get food marks.

5) Consider a predator-prey system of the form:

$$\begin{cases} \frac{dx}{dt} &= x\left(1 - 1/2y - 1/4x\right) \\ \frac{dy}{dt} &= y\left(-1 + 1/2x\right) \end{cases} \tag{3}$$

Here $x(t) \geq 0$ models the population of prey in an environment and $y(t) \geq 0$ models the population of predators in the same environment. Note the difference from classic Lotka-Volterra system: we have added a term proportional to $-1/4x^2$ to the first equation in (3) to model the competition between prey for a limited amount of food.

5)a) There are exactly three equilibrium points for the system (3). Find them

Solution: Clearly we can set $x = y = 0$ to find one equilibrium point $(0, 0)$. Another is found by setting $y = 0$ and $1 - \frac{1}{4}x = 0$ in other words $(4, 0)$. There are no points with $x = 0$ and $y \neq 0$ (the condition $x = 0$ forces $y = 0$ as well)

Finally we look for point with $x \neq 0$ and $y \neq 0$. In this case we have

$$\begin{cases} -1 + \frac{1}{2}x &= 0 \\ 1 - 1/2y - 1/4x &= 0 \end{cases}$$

that is easily solved by $x = 2$ and $y = 1$. So the final equilibrium point is $(2, 1)$.

5)b) Show that the equilibrium points $(0, 0)$ and $(4, 0)$ are unstable saddle point. Find the eigenvalues and eigenvectors of a linearisation of (3) at these points.

Solution the point $(0, 0)$ represents the point where all the rabbits and foxes are dead. Looking for the jacobian at this point, we have

$$J(x, y) = \begin{pmatrix} 1 - \frac{1}{2}y - \frac{1}{2}x & -\frac{1}{2}x \\ \frac{1}{2}y & -1 + \frac{1}{2}x \end{pmatrix} \tag{4}$$

Thus at $(0, 0)$, the Jacobian is

$$J(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This matrix is diagonal and the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$. The corresponding vectors lie along the axes that is the solution to the equation:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = J(0, 0)\begin{pmatrix} x \\ y \end{pmatrix} \tag{5}$$

is $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$. Note of course that in writing (5), we neglect the higher order terms in the full nonlinear equation. So this solution is really valid only close to the origin.
Think about the physical interpretation as well. The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents the x-axis where there are no foxes so rabbits are happily reproducing at an exponential rate.
On the other hand, the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents the y-axis where there are no rabbits and the foxes are dying of starvation at an exponential rate.
Moving on to the other point, $(4, 0)$ we have

$$J(4,0) = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} \tag{6}$$

Solving for the eigenvalues:

$$\begin{aligned} \det\left(J(4,0) - \lambda I\right) &= \begin{pmatrix} -1-\lambda & -2 \\ 0 & 1-\lambda \end{pmatrix} \\ &= (-1-\lambda)(1-\lambda) = 0 \end{aligned}$$

So again $\lambda_1 = 1$ and $\lambda_2 = -1$ are the eigenvalues. Searching for the eigenvector $\xi^{(1)}$ corresponding to λ_1 involves solving

$$\begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\begin{cases} -\xi_1 - 2\xi_2 & = \xi_1 \\ \xi_2 & = \xi_2 \end{cases}$$

The second equation tells us nothing but the first can be rewritten as $\xi_1 + \xi_2 = 0$ therefore we choose $\xi^{(0)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
For the second eigenvector $\xi^{(2)}$ corresponding to λ_2 we solve

$$\begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = -\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\begin{cases} -\xi_1 + -2\xi_2 & = -\xi_1 \\ \xi_2 = -\xi^2 \end{cases}$$

both of which are telling us $\xi_2 = 0$ and so $\xi^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ So there is a sadlle point near to $(4, 0)$. We write the solution near to this equilibrium point by translating the variables x, y to $\tilde{x} = x - 4$ (note we don't need to change y). We thus have

$$\frac{d}{dt}\begin{pmatrix} \tilde{x} \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1// - 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$$

The second vector again corresponds to no predators however it now comes with a negative eigenvalue meaning the population of rabbits will decrease even if it is larger than 4 even in the absence of predators. This can be explained by the extra factor we added to the Lotka-Volterra model. If you set $y = 0$, the equation for x is $x' = x(1 - \frac{1}{4}x$ that is a typical autnomous system for one dependent variable. All we've found is that $x = 0$ is unstable and $x = 4$ is stable.

The other eigenvector points in a sensible direction: an increase of foxes and a decrease of rabbits.

5)c) Show that the linearisation at other equilibrium point is a stable spiral

Solution: here we have the Jacobian:

$$J(2, 1) = \begin{pmatrix} -1/2 & -1 \\ 1/2 & 0 \end{pmatrix}$$

Finding the eigenvalues of $J(2, 1)$, we have

$$\begin{aligned} \det\left(J(2,1) - \lambda I\right) &= \begin{pmatrix} -1/2 - \lambda & -1 \\ 1/2 & -\lambda \end{pmatrix} \\ &= \left(-1/2 - \lambda\right)\left(-\lambda\right) + 1/2 \\ &= \lambda^2 + 1/2\lambda + 1/2 = 0 \end{aligned}$$

Using the quadratic equation, we get

$$\begin{aligned} \lambda &= 1/2\left(-1/2 \pm \sqrt{1/4 - 2}\right) \\ &= 1/2\left(-1/2 \pm i\sqrt{7/4}\right) \end{aligned}$$

Since the eigenvalues are complex conjugates and the real part is neagative, this is a stable spiral.

5)d) Draw an approximate sketch of the full nonlinear solution to this system. (make sure it includes your answers to b) and c)). What conclusion can you propose about the behaviour as $t \to \infty$ of $x(t)$ and $y(t)$ so long as both are strictly positive?

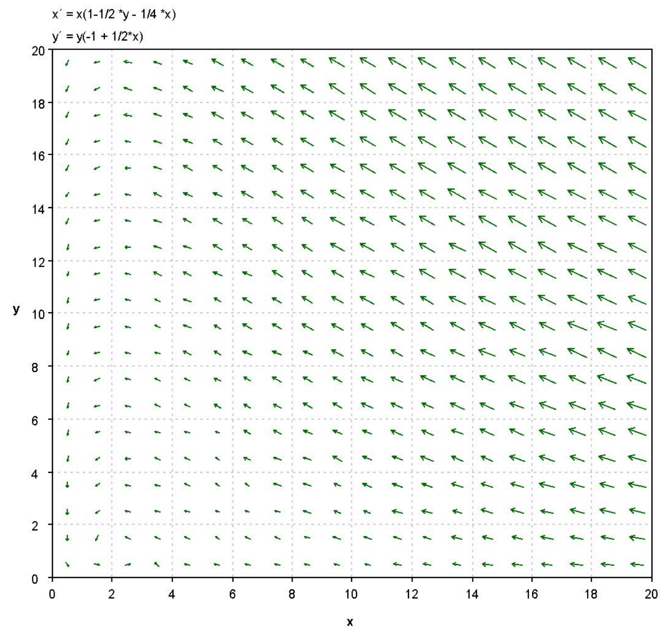

Figure 7: A direction field for the system of equations (3)

5)e) If you have access to software which can approximately solve these equations, verify numerically that your guess is correct.

A simple java applet is available at

`http://math.rice.edu/~dfield/dfpp.html` that will draw the direction field of any nonlinear system.

For part d) anything reasonably drawn gets full marks. for e), we can use the applet to draw the phase plane, see Figure 7 for example

The main point is that the equilibrium point $(2, 1)$ is a very strong attractor. No matter where we start, so long as $x(0) > 0$ and $y(0) > 0$, we have $\lim_{t\to\infty} (x(t), y(t)) = (2, 1)$.