Answer the following questions. Each is worth 20 points for a total of 100.

1. **The Linear Problem**

   Consider then linear differential equation for $y(t)$

$$y'(t) - \frac{t}{1-t^2}y = t$$

   (a) Find the general solution to this equation (using integrating factors is helpful)

   Solution: We use the integrating factor with $p(t) = \frac{-t}{1-t^2}$. Since (using the substitution $u = 1 - t^2$)

$$\begin{aligned}\int \frac{-t}{1-t^2} &= \frac{1}{2}\int \frac{du}{u} \\ &= \frac{1}{2}\ln|u| \\ &= \ln\left|1-t^2\right|^{1/2}\end{aligned}$$

   we have $\mu(t) = \exp\left(\int p\right) = |1-t^2|^{1/2}$. Multiplying both sides of the equation by μ gives

$$(\mu y)' = \left|1-t^2\right|^{1/2} t$$

   The absolute value sign depends on if whether $|t| < 1$ or $|t| > 1$.

   Case 1: $|t| < 1$ in this case, suppose we integrate between t_0 and t with both $|t_0|, |t| < 1$, then the fundamental theorem of calculus gives

$$\begin{aligned}(1-t^2)^{1/2}y(t) - (1-t_0)^{1/2}y(t_0) &= \int_{t_0}^{t}(1-s^2)^{1/2}sds \\ &= -\frac{1}{2}\int u^{1/2}du \\ &= -\frac{1}{3}u^{3/2} \\ &= -\frac{1}{3}\left\{\left(1-t^2\right)^{3/2} - \left(1-t_0^2\right)^{3/2}\right\}\end{aligned}$$

   Since t_0 is arbitrary with magnitude less than one, the general solution is

$$\begin{aligned}\sqrt{1-t^2}y(t) &= -\frac{1}{3}\left(1-t^2\right)^{3/2} + c \\ y(t) &= -\frac{1}{3}\left(1-t^2\right) + c\left(1-t^2\right)^{-1/2} \qquad (1)\end{aligned}$$

where c depends on the initial condition of the system.

Case 2: $|t| > 1$ There should really be 2 subcases but let's consider only if $t > 1$ and $t_0 > 1$. We have $\mu(s) = (s^2 - 1)^{1/2}$ for any $s > 1$ and so we write

$$\begin{aligned}\left(t^2 - 1\right)^{1/2} y(t) - (t_0^2 - 1)^{1/2} y(t_0) &= \int_{t_0}^{t} \left(s^2 - 1\right)^{1/2} s ds \\ &= \frac{1}{2} \int u^{1/2} du \\ &= \frac{1}{3} \left\{ \left(t^2 - 1\right)^{3/2} - \left(t_0^2 - 1\right)^{3/2} \right\}\end{aligned}$$

where we use now $u = s^2 - 1$ for the substitution. Since the starting point t_0 is arbitrary (except that $t_0 > 1$) we have that

$$y(t) = \frac{1}{3} \left(t^2 - 1\right) + c \left(t^2 - 1\right)^{-1/2} \tag{2}$$

The case where $t_0 < -1$ and $t < -1$ is handled similarly.

The main point to take away from these calculations is that the solutions will be different for $|t| < 1$ and $|t| > 1$.

Can we integrate from t_0 to t when $|t_0| < 1$ but $|t| > 1$? that's a good question

(b) Find the particular solution corresponding to the initial condition

$$y(t = 0) = 2$$

Since $t_0 = 0$ in this case, we set $\mu(t) = 1 - t^2$ and use the first solution above:

$$y(t) = -\frac{1}{3} \left(1 - t^2\right) + c \left(1 - t^2\right)^{-1/2}$$

setting $y(0) = 2$ gives an equation for c:

$$y(0) = -\frac{1}{3} + c = 2 \implies c = \frac{7}{3}$$

Thuse the particular solution corresponding to $y(0) = 2$ is

$$y(t) = -\frac{1}{3} \left(1 - t^2\right) + \frac{7}{3} \left(1 - t^2\right)^{-1/2}$$

(c) On which interval of values of t is the solution in part b) valid?

Solution: Clearly the right hand side makes sense only if $t \neq \pm 1$. So the interval of validity is $t \in (-1, 1)$. You could have expected this by examining when the function $p(t) = \frac{-t}{1-t^2}$ is continuous.

2. **The Fishery Problem** Suppose that you operate a salmon farm where your fish reproduce at a natural rate with the logistic model with rate r and optimal population P_∞. Without external factors the population P obeys the standard equation :

$$\frac{dP}{dt} = r\left(1 - \frac{P}{P_\infty}\right)P$$

. This being a farm, you harvest the fish at a rate A proportional to its population. So $0 < A < r$ (you don't want to eat the fish faster than they can reproduce). Thus the equation for the population becomes

$$\frac{dP}{dt} = r\left(1 - P/P_\infty\right)P - AP$$

(a) Show that $P^1_{EQ} = 0$ and $P^2_{EQ} = P_\infty\left(1 - \frac{A}{r}\right)$ are equilibrium points. Show that P^1_{EQ} is unstable while P^2_{EQ} is stable. Include a diagram with a few solution curves.

Solution:

For an autonomous system like this one, the equilibrium points are the zeros of the function

$$\begin{aligned} f(P) &= r\left(1 - \frac{P}{P_\infty}\right)P - AP \\ &= rP - \frac{rP}{P_\infty} - AP \\ &= P\left(r - A - r\frac{P}{P_\infty}\right) \end{aligned}$$

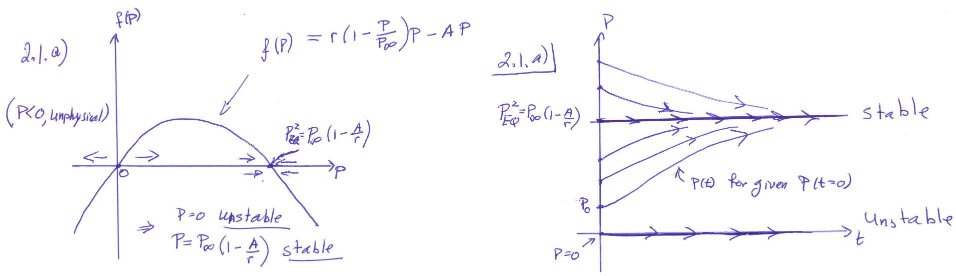
So either $P = 0$ or $P = P_\infty\left(1 - \frac{A}{r}\right)$. To determine stability, note that we can write

$$f(P) = -\frac{r}{P_\infty}P\left(P - \frac{P_\infty}{r}(r - A)\right)$$

Here $r > 0$ and $P_\infty > 0$ are parameters. If $P < 0$ then both terms are negative and $f(P) < 0$ (this case is not physical!).

If $0 < P < \frac{P_\infty}{r}(r - A)$, then $f(P) > 0$. By definition, $P = 0$ is unstable.

Finally, if $P > \frac{P_\infty}{r}(r - A)$ then $f(P) < 0$ and by definition $P = \frac{P_\infty}{r}(r - A)$ is stable



(a) A sketch of $f(P)$ showing $P = 0$ is unstable while P^2_{EQ} is stable

(b) Some solution curves from various values of P_0

Figure 1: Sketch of the situation in problem 2

(b) Suppose you want to maximize the yield of fish you eat while keeping the population stable near the value P^2_{EQ}. What value of the catching rate A optimizes the yield $Y = AP$? What percentage of fish population do you eat per year at the optimal rate?

Solution: Keeping P stable around the stable point $P = \frac{P_\infty}{r}(r - A)$, we maximize

$$\begin{aligned} Y &= AP \\ &= \frac{P_\infty}{r}\left(rA - A^2\right) \end{aligned}$$

To maximize, we set $\frac{dY}{dA} = 0$ and solve for A:

$$\frac{dY}{dA} = \frac{P_\infty}{r}(r - 2A) = 0 \implies A = \frac{r}{2}$$

The maximum yield at this rate of fishing is

$$Y(\frac{r}{2}) = \frac{P_\infty}{r}\left(\frac{r^2}{2} - \frac{r^2}{4}\right) = \frac{rP_\infty}{4}$$

Therefore if the rate is to be measured in fish per year, we eat a quarter of the fish

3. **The Red Tide attacks the fishery** Now an unfortunate event happens and a bloom of algae called a red tide `https://en.wikipedia.org/wiki/Red_tide` poisons a number of your fish. The algae poison your fish at a rate $R > 0$ independent of the current population. The equation for the fish population is now

$$\frac{dP}{dt} = r\left(1 - \frac{P}{P_\infty}\right)P - R - AP$$

(a) Show that there is a critical rate of poisoning $R_{crit} = \frac{P_\infty}{4r}(r-A)^2$ so that if $R < R_{crit}$ there are still equilibrium solutions $P^1_{EQ} < P^2_{EQ}$ such that P^1_{EQ} is unstable while P^2_{EQ} is stable. Include a diagram that shows a few solution curves.

Solution: This is again an autonomous system with

$$f(P) = r\left(1 - \frac{P}{P_\infty}\right)P - R - AP$$

Searching for equilibrium points equates to finding points with $f(P) = 0$. This can be written as a quadratic formula:

$$-f(P) = \frac{r}{P_\infty}P^2 - (r-A)P + R = 0$$

The quadratic formula gives the equilibrium solutions:

$$P_{EQ} = \frac{P_\infty}{2r}(r-A) \pm \frac{P_\infty}{2r}\sqrt{(r-A)^2 - 4\frac{rR}{P_\infty}}$$

For the argument of the square root to be nonnegative, we need that

$$(r-A)^2 - 4\frac{rR}{P_\infty} \geq 0$$
$$R \leq \frac{P_\infty}{4r}(r-A)^2$$

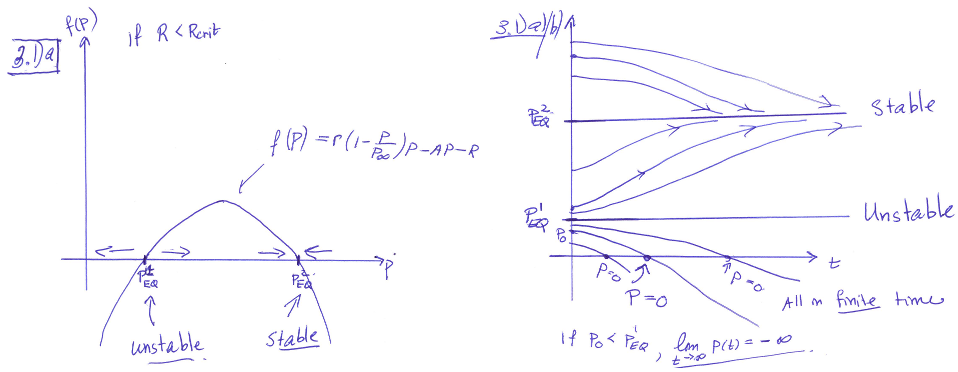
Defining $R_{crit} = \frac{P_\infty}{4r}(r-A)^2$, we see that there are no equilibrium solutions if $R > R_{crit}$ but there are two distinct solutions if $R < R_{crit}$. These are

$$P^1_{EQ} = \frac{P_\infty}{2r}(r-A) - \frac{P_\infty}{2r}\sqrt{(r-A)^2 - 4\frac{rR}{P_\infty}}$$
$$P^2_{EQ} = \frac{P_\infty}{2r}(r-A) + \frac{P_\infty}{2r}\sqrt{(r-A)^2 - 4\frac{rR}{P_\infty}}$$

Since $f(P)$ is a parabola that opens downwards, it is easy to check as in part 2)b) that P^1_{EQ} is unstable while P^2_{EQ} is stable.

(b) Show that $P^1_{EQ} > 0$. Suppose that you initially had a small number of fish $P(t=0) = P_0 < P^1_{EQ}$. Prove that all your fish die in **finite** time. (A diagram might help)

We write the formula for P^1_{EQ} as



(a) A sketch of $f(P)$ showing P^1_{EQ} is unstable while P^2_{EQ} is stable

(b) Some solution curves from various values of P_0

Figure 2: Sketch of the situation in problem 3

$$
\begin{aligned}
P^1_{EQ} &= \frac{P_\infty}{2r}(r-A) - \frac{P_\infty}{2r}\sqrt{(r-A)^2 - 4\frac{rR}{P_\infty}} \\
&= \frac{P_\infty}{2r}(r-A)\left\{1 - \sqrt{1 - 4\frac{rR}{P_\infty(r-A)^2}}\right\}
\end{aligned}
$$

Since $4\frac{rR}{P_\infty(r-A)^2} > 0$, we can conclude that $P^1_{EQ} > 0$.
If $P(t=0) < P^1_{EQ}$ then it is clear from the diagrams that $\frac{dP}{dt} < 0$ for every $t > 0$. Thus $\lim_{t\to\infty} P = -\infty$ and there must be some $t_F \in (0,\infty)$ with $P(t_F) = 0$. See Figure 2 (b)

(c) Typically the food demands remain the same and people continue to fish at the same rate they did in normal circumstance. Suppose you continue to catch fish at the rate you obtained in problem 2)b). Prove that if the poisoning rate gets too strong, namely $R > \frac{P_\infty r}{16}$, all your fish die in **finite** time no matter how many you had at the start. (Again a diagram might help)

Solution: One way to approach this problem is to find the value of R for which there are no equilibrium solutions at all and $f(P) < 0$ for any value of P. Taking the fishing rate $A = \frac{r}{2}$ from 2)b), we see that such a value is R_{crit} evaluated at this fishing rate:

$$
\begin{aligned}
R_{crit} &= \frac{P_\infty}{4r}(r-A)^2 \\
&= \frac{P_\infty}{4r}\left(\frac{r}{2}\right)^2 \\
&= \frac{rP_\infty}{16}
\end{aligned}
$$

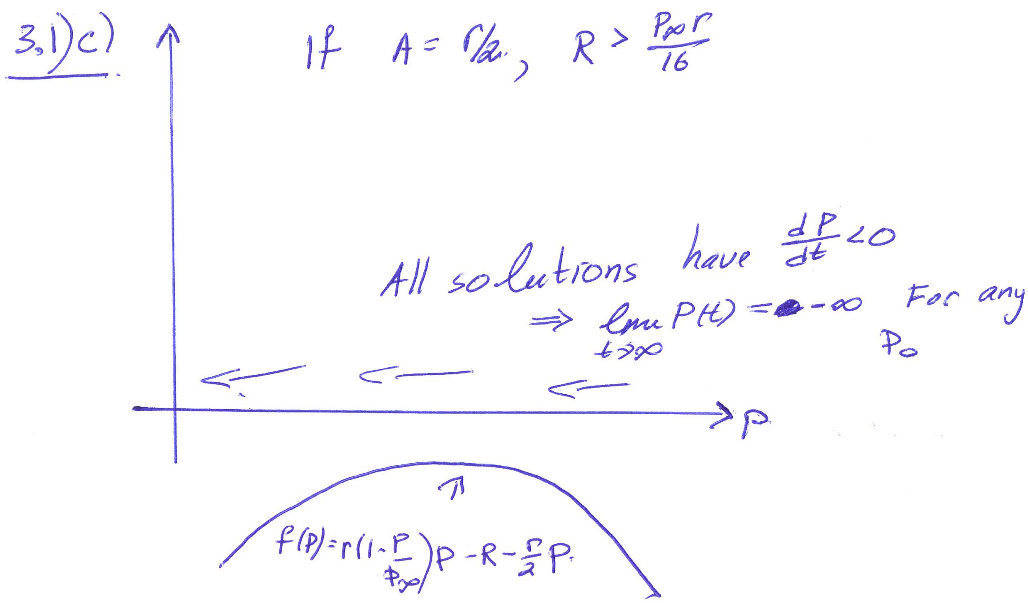


Figure 3: When $R > R_{crit}$, $f(P) < 0$ for all values of P. Thus $\frac{dP}{dt} < 0$ and $\lim_{t\to\infty} P(t) = -\infty$ for any initial value P_0. All your fish will die.

In this case $f(P) < 0$ for any P and we have that for any value $P(t = 0) = P_0 > 0$, $\frac{dP}{dt} < 0$ for all $t > 0$

$$\lim_{t\to\infty} P(t) = -\infty$$

and there must be some finite time t_F with $P(t_F) = 0$. See diagram in Figure 3

Incidentally, the fishermen protesting and blocking roads due to an ongoing event of red tide in southern Chile is the reason we started the course a lecture late.

4. **The Exact Problem**

(a) Solve the following equation for $y(t)$. Leave the solution in implicit form:

$$\frac{dy}{dt} = \frac{-2 - ye^{ty}}{-2y + te^{ty}}$$

write this equation as

$$2 + ye^{ty} + \left(-2y + te^{ty}\right)\frac{dy}{dt} = 0.$$

Let $M(x, y) = 2 + ye^{ty}$ and $N(x, y) = -2y + te^{ty}$, we observe that

$$\frac{\partial M}{\partial y} = (1 + ty)\, e^{ty}$$
$$\frac{\partial N}{\partial t} = (1 + ty)\, e^{ty}.$$

Thus this equation is exact and its solutions $y(t)$ sketch out a part of a level curve of some multivariable function $\psi(x,t)$. We integrate to find ψ:

$$\begin{aligned}\psi(x,t) &= \int \left(2 + ye^{ty}\right) dt \\ &= 2t + e^{ty} + f(y)\end{aligned}$$

where $f(y)$ is to be determined. We check that ψ_y corresponds to $N(x,y)$. On the one hand

$$\psi_y = te^{ty} + f'(y)$$

while

$$N = -2y + te^{ty}$$

Thus $f'(y) = -2y$ or $f(y) = -y^2$. Finally, $y(t)$ solves

$$2t - y^2 + e^{ty} = c$$

for some constant c that depends on the initial conditions.

(b) Check if the following equation for $y(x)$ is exact:

$$y + \left(3xy - e^{-3y}\right) \frac{dy}{dx} = 0.$$

If it is not exact, find an integrating factor that makes it exact and solve for $y(x)$. You can leave the solution in implicit form. (hint: multiply both sides of the equation by an appropriate function $\mu(y)$.)

In this case $M(y,x) = y$ while $N(y,x) = 3xy - e^{-3y}$. We check that this equation is not exact

$$\frac{\partial M}{\partial y} = 1$$
$$\frac{\partial N}{\partial x} = 3y.$$

Therefore it makes no sense to look for a function $\psi(xy)$ with $\psi_{xy} \neq \psi_{yx}$. We try to make it exact by multiplying both sides by some $\mu(y)$:

$$y\mu + \left(3xy\mu - e^{-3y}\mu\right)\frac{dy}{dx} = 0.$$

For this to be exact, we equate

$$\begin{aligned} (y\mu)_y &= \mu + y\mu' \\ \left(3xy\mu - e^{-3y}\mu\right)_x &= 3y\mu \end{aligned} \tag{3}$$

Thus $\mu(y)$ solves the simple equation

$$\begin{aligned} \mu + y\mu' &= 3y\mu \\ \mu' &= \frac{1}{y}\left(-\mu + 3y\mu\right) \\ \frac{\mu'}{\mu} &= \frac{-1}{y} + 3 \end{aligned}$$

Integrating both sides with respect to y gives

$$\begin{aligned} \ln|\mu| &= 3y - \ln|y| \\ |\mu| &= \frac{1}{|y|}e^{3y}. \end{aligned}$$

Unlike question 1)a), the absolute value matters not at all, it amounts to multiplying the whole equation by $+1$ or -1 which can't affect solutions. We choose $\mu(y) = \frac{1}{y}e^{3y}$. Our equation now becomes

$$e^{3y} + \left(3xe^{3y} - \frac{1}{y}\right)\frac{dy}{dx} = 0$$

A quick sanity check:

$$\begin{aligned} \frac{\partial}{\partial y}\left(e^{3y}\right) &= 3e^{3y} \\ \frac{\partial}{\partial x}\left(3xe^{3y} - \frac{1}{y}\right) &= 3e^{3y} \end{aligned}$$

So this guy is exact! We solve

$$\psi = \int e^{3y} dx = xe^{3y} + f(y)$$

where $f(y)$ is to be determined. On the one hand

$$\psi_y = 3xe^{3y} + f'(y).$$

On the other hand, from the equation

$$\psi_y = 3xe^{3y} - \frac{1}{y}.$$

Therefore $f'(y) = -\frac{1}{y}$ or $f(y) = \ln \frac{1}{y}$. To conclude, $y(x)$ solves

$$xe^{3y} + \ln \frac{1}{y} = c$$

for some constant c depending on the initial conditions.

5. **The Uniqueness Problem** (Don't be scared by the many parts, only e) is challenging.)

Consider the nonlinear differential equation

$$y'(x) = x \arcsin(y) \tag{4}$$

Note: you may have seen $\arcsin(y)$ denoted as $\sin^{-1}(y)$ before. It is simply the inverse function to the sine restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(a) Show that given the initial condition $y(x=0) = y_0 = 0$, then $y(x) = 0$ is a solution to (4) on its domain

If $y(x) = 0$ then $y'(x) = 0$ for any x while $x \arcsin(y(x)) = x \arcsin 0 = 0$ for any x and this function solves the differential equation.

(b) Suppose that there is another solution $y_2(x)$ to (4) with $y_2(x=0) = 0$ but $y_2(x) \neq 0$ for some other value of x. Because y_2 is continuous, show there is some $s > 0$ so that $|y_2(x)| < \frac{1}{2}$ whenever $x \in [-s, s]$.

Since y_2 is continuous, $\lim_{x \to 0} y_2(x) = 0$. Thus the values tend to 0 and you can find an interval around 0 where $|y_2(x)| \leq \frac{1}{2}$.

Precisely: given $\epsilon = \frac{1}{2}$, there is a $\delta > 0$ so that $|y_2(x)| < \frac{1}{2}$ whenever $x \in (-\delta, \delta)$. Then $s = \frac{\delta}{2}$ works.

(c) Using the mean value theorem (and the fact that $|y_2| < \frac{1}{2}$ for these x values!), show that $\arcsin(y_2(x)) \leq \frac{2}{\sqrt{3}} y_2(x)$ for any $x \in [-s, s]$

Applying the MVT to $\arcsin(y)$ between y_1 and y_2, we have

$$\frac{d}{dy}\arcsin(y)\,\Big|_{y=y^*} = \frac{\arcsin\left(y_2(x)\right) - \arcsin\left(y_1(x)\right)}{y_2(x) - y_1(x)} \tag{5}$$

for some $y^*(x)$ between $y_1(x)$ and $y_2(x)$. Now $y_1(x) = 0$ for all x, and

$$\frac{d}{dy}\arcsin(y)\,\Big|_{y=y^*} = \frac{1}{\sqrt{1 - [y^*(x)]^2}}$$

Since $y^*(x)$ is between $y_1(x) = 0$ and $|y_2(x)| \leq \frac{1}{2}$, we conclude that

$$\begin{aligned}\frac{1}{\sqrt{1 - [y^*(x)]^2}} &\leq \frac{1}{\sqrt{1 - \left(\frac{1}{2}\right)^2}} \\ &= \frac{1}{\sqrt{1 - \frac{1}{4}}} \\ &= \frac{2}{\sqrt{3}}\end{aligned}$$

Returning to the mean value theorem (5), we have

$$\begin{aligned}\frac{2}{\sqrt{3}} &\geq \frac{\arcsin\left(y_2(x)\right) - \arcsin\left(y_1(x)\right)}{y_2(x) - y_1(x)} \\ &= \frac{\arcsin\left(y_2(x)\right)}{y_2(x)}\end{aligned}$$

Therefore $\arcsin\left(y_2(x)\right) \leq \frac{2}{\sqrt{3}} y_2(x)$ as needed.

(d) Applying the fundamental theorem of calculus, show that for any $x \in [-s, s]$

$$\begin{aligned}y_2(x) &= \int_0^x \tilde{x}\arcsin(y_2(\tilde{x}))d\tilde{x} \\ &\leq \frac{2s}{\sqrt{3}} \int_0^x |y_2(\tilde{x})| d\tilde{x}\end{aligned}$$

From the ODE, $y_2'(x) = x\arcsin\left(y_2(x)\right)$, we integrate both sides between 0 and some $x \in [-s, s]$:

$$\int_0^x y_2'(tx)dtx = \int_0^x \tilde{x} \arcsin\left(y_2(\tilde{x})\right) d\tilde{x}$$

On the left hand side,

$$\begin{aligned}\int_0^x y_2'(tx)dtx &= y_2(x) - y_2(0) \\ &= y_2(x)\end{aligned}$$

On the right hand side,

$$\begin{aligned}\int_0^x \tilde{x} \arcsin\left(y_2(\tilde{x})\right) d\tilde{x} &\leq \left| \int_0^x \tilde{x} \arcsin\left(y_2(\tilde{x})\right) d\tilde{x} \right| \\ &\leq \int_0^x |\tilde{x}| |\arcsin\left(y_2(\tilde{x})\right)| d\tilde{x} \\ &\leq \frac{2}{\sqrt{3}} |x| \int_0^x |y_2(\tilde{x})| d\tilde{x} \\ &\leq \frac{2s}{\sqrt{3}} \int |y_2(\tilde{x})| d\tilde{x}\end{aligned}$$

The last step follows from the fact that $|x| \leq s$ and the penultimate step from part (c)

(e) Let $F(x) = \int_0^x |y_2(\tilde{x})| d\tilde{x}$. By part d), we have that $F'(x) \leq \frac{2s}{\sqrt{3}} F(x)$. Using that $F(0) = 0$, show $F(x) = 0$ for every $x \in [-s, s]$ and thus $y_2(x) = 0$ as well by continuity (hint: Look up Gronwall's lemma).

Solution: Defining $F(x) = \int_0^x |y_2(\tilde{x})| d\tilde{x}$, the second part of the FTC tells us $F'(x) = |y_2(x)|$. Thus by part (d), we have a differential inequality:

$$F'(x) \leq \frac{2s}{\sqrt{3}} F(x)$$

and moreover $F(0) = 0$. Differential inequalities are NOT ODEs. But we know what would happen if I replaced $\leq$ by $=$ above.

$$\begin{cases} F'(x) = \frac{2s}{\sqrt{3}} F(x) \\ F(0) = 0 \end{cases} \implies F(x) = F(0) e^{\frac{2s}{\sqrt{3}} x} = 0 \cdot e^{\frac{2s}{\sqrt{3}} x} = 0$$

Now suppose that $x > 0$, then $F(x) \geq 0$ for any x (we integrate an absolute value). Defining

$$v(x) = F(x)e^{-\frac{2s}{\sqrt{3}}x}$$

we also have $v(x) \geq 0$. Compute

$$\begin{aligned} v'(x) &= \left(F'(x) - \frac{2s}{\sqrt{3}}F(x)\right)e^{-\frac{2s}{\sqrt{3}}x} \\ &\leq \left(\frac{2s}{\sqrt{3}}F(x) - \frac{2s}{\sqrt{3}}F(x)\right)e^{-\frac{2s}{\sqrt{3}}x} \\ &= 0 \end{aligned} \tag{6}$$

So $v(0) = 0$, $v(x) \geq 0$ and $v'(x) \leq 0$ for any x. It must be that $v(x) = 0$ for every x. Since exponentials are not zero, we must have $F(x) = 0$ for every x.

Now if $x < 0$, $F(x) \leq 0$ so this trick doesn't go as smoothly. Instead we define $\tilde{z} = -\tilde{x}$ and write

$$F(x) = -\int_0^{-x} |y_2(-\tilde{z})d\tilde{z}$$

If $x < 0$, then $-x > 0$ and the calculation in (6) applied to

$$\int_0^{-x} |y_2(-\tilde{z})|d\tilde{z}$$

Shows that $F(x) = 0$ even for $x < 0$.

Finally suppose we did have non-uniqueness and a point $x_0 \in [-s, s]$ with $y_2(x_0) = A > 0$ say. Since y_2 is continuous, it doesn't just jump from 0 to A. There must be an interval where $y_2(x) > \frac{1}{2}A$ first (say $y_2(x) > \frac{1}{2}A$ when $x \in (x_0 - \delta, x_0 + \delta)$. Then clearly, we'd have

$$\begin{aligned} F(s) &= \int_0^s |y_2(\tilde{x})|d\tilde{x} \\ &\geq \int_{x_0-\delta}^{x_0+\delta} |y_2(\tilde{x})|d\tilde{x} \\ &\geq \frac{1}{2}A\delta \end{aligned}$$

But we just proved that $F(x) = 0$ for any $x \in [-s, s]$ so this is a contradiction!

Therefore, at least for $x \in [-s, s]$ the solution $y(x) = 0$ for all x is unique

NOTE: You don't have to be this precise to get full marks but it's nice to see all the details sometimes.

Therefore the solution $y(x) = 0$ for every x is unique near to 0!

(f) Why will this argument fail if I had said $y(x = 0) = y_0 = 1$ in part a)

Solution. The main reason is that we wouldn't be able to bound

$$\frac{d}{dy} \arcsin(y) \,|_{y=y^*} = \frac{1}{\sqrt{1 - [y^*(x)]^2}}$$

if $y^*(x)$ could take any value close to 1