Tutorial 4 - MAT244 - C.J. Adkins

Exact Eq & Euler's Method

Need functions M, N, M_y, N_x to be continuous in some open box $= (\alpha,\beta)\times(\gamma,\delta)$, then

$M + Ny' = 0$ has a solution $\iff M_y = N_x$ (No poles! (Blow ups))

Why? Rewrite in differential form:

$$M\,dx + N\,dy = 0 \iff d\big(F(x,y)\big) = d(\text{const})$$ This is called "exact"

If M, N, M_y, N_x are continuous we can integrate to find $F(x,y)$, the solution.

Remark: This is a result of something called the exterior differential "d", it satisfies $d^2 = 0$

Ex (2.6-#11) Is the O.D.E exact? if so, solve!

$$(\overbrace{x\ln(y)+xy}^{M}) + (\overbrace{y\ln(x)+xy}^{N})y' = 0 \quad , x>0$$

$\Rightarrow M_y = \frac{x}{y} + x \;,\; N_x = \frac{y}{x} + y \;, \Rightarrow M_y \neq N_x \Rightarrow$ Not Exact

Ex (2.6-#12) Is the O.D.E exact? If so, solve!

$$\overbrace{\frac{x}{(x^2+y^2)^{3/2}}}^{M} + \overbrace{\frac{y}{(x^2+y^2)^{3/2}}}^{N}\frac{dy}{dx} = 0$$

$\Rightarrow M_y = \frac{-3xy}{(x^2+y^2)^{5/2}} = N_x \Rightarrow$ Exact!

use $u = x^2+y^2$, $du = 2x\,dx$; $v = x^2+y^2$, $dv = 2y\,dy$

How to solve? $F(x,y) = \int M\,dx + \int N\,dy = \int \frac{x\,dx}{(x^2+y^2)^{3/2}} + \int \frac{y\,dy}{(x^2+y^2)^{3/2}} = \frac{1}{2}\left[\int \frac{du}{u^{3/2}} + \int \frac{dv}{v^{3/2}}\right]$

Therefore the solution is $F(x,y) = \frac{-1}{2}\left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{v}}\right) = \boxed{\frac{-1}{\sqrt{x^2+y^2}} = \text{const}}$

Ex (2.6-#21) Show the O.D.E is not Exact but can be with an integrating factor

$*\quad \underbrace{y}_{M} + (\overbrace{2x - ye^y}^{N})y' = 0 \;,\; \mu(x,y) = y$

Notice that $M_y = 1 \neq N_x = 2 \Rightarrow$ Not exact! But for $\mu *$ we have

$$y^2 + (2xy - y^2e^y)y' = 0$$

which gives $M_y = 2y = 2y = N_x \checkmark \Rightarrow$ Exact, We solve By $F(x,y) = \int M\,dx + \int N\,dy$

repeated)

$$= \int y^2 dx + \int (2xy - y^2 e^y) dy = y^2 x + xy^2 - (y^2 - 2y + 2) e^y$$

$$\Rightarrow \text{Const} = xy^2 - e^y(y^2 - 2y + 2) \text{ is the solution.}$$

Finding the integration constant!

Ex (2.6 - #24) If $(N_x - M_y)/(xM - yN) = R(xy)$ then find the integrating factor to

$$M + Ny' = 0$$

Well, what do we want? $\mu M + \mu N y' = 0$ to be exact with $\mu(xy)$, so...

$(\mu M)_y = xM\mu' + \mu M_y$ & $(\mu N)_x = yM\mu' + \mu N_x$ ... want them to be =, thus

$$xM\mu' + \mu M_y = yM\mu' + \mu N_x \iff (xM - yN)\mu' = \mu(N_x - M_y)$$

$$\iff \frac{\mu'}{\mu} = \frac{N_x - M_y}{xM - yN} = R \iff \mu(u) = \exp\left(\int R(u)\, du\right) \quad \text{“}u = xy\text{''}$$

Question: How to get $\mu(x)$, $\mu(y)$ from this

In general, we're just trying to find something to make it exact.

Euler's Method (1st Order Taylor)

Recall the definition of slope $\frac{\Delta y}{\Delta x} = \frac{y_{n+1} - y_n}{x_{n+1} - x_n} \approx y'$ if $x_{n+1} - x_n$ is small

Define: $x_n = \varepsilon + x_{n-1}$ w/ x_0 as the first point (ε is the step size, usually small)

then if we had $y' = f(x,y)$ we know that

$$\Rightarrow y(x_{n+1}) - y(x_n) \approx \underbrace{(x_{n+1} - x_n)}_{\Delta x = \varepsilon} f(x_n, y_n) \Rightarrow y(x_{n+1}) \approx y(x_n) + \varepsilon f(x_n, y_n)$$

If we have some initial data $x_0, y(x_0)$ then we know the approximate solution by iterations!

Ex Approximate $y' = f(x,y)$

$$y(x_0) = x_0$$
$$y(x_1) = y(x_0) + \varepsilon f(x_0, y_0)$$
$$y(x_2) = y(x_1) + \varepsilon f(x_1, y_1)$$
$$\vdots \text{ etc.}$$

Note: Book uses h as step size, not ε.

Why does this work? Direction fields! I.e we glue together slope lines to approx solution! So as long as f & $\frac{\partial f}{\partial y}$ are continuous (or Lipschitz) we're fine!

Ex (2.7 - #20) (convergence of Euler's Method)

Consider: $y' = 1 - t + y$, $y(t_0) = y_0$

a) By solving explicitly (first order linear) we know that

$$y(t) = (y_0 - t_0)e^{t - t_0} + t$$

b) If we use Euler's Method, we see that

$$y(t_{n+1}) = y(t_n) + \varepsilon f(t_n, y_n) = y(t_n) + \varepsilon(1 - t_n + y_n)$$
$$= (1+\varepsilon)y(t_n) + \varepsilon(1 - t_n) \;*$$

c) If we use the definition recursively, we can get everything in terms of initial data
I.e. By induction with $y(t_1) = (1+\varepsilon)(y(t_0) - t_0) + t_1$ we can show

$$y(t_n) = (1+\varepsilon)^n (y(t_0) - t_0) + t_n$$

check base case ✓ Assume true for n & prove n+1 $\quad t_{n+1} = \varepsilon + t_n$

$$y(t_{n+1}) = (1+\varepsilon)^{n+1}(y(t_0) - t_0) + t_{n+1} = \underbrace{(1+\varepsilon)^n (y(t_0) - t_0)} + (1+\varepsilon)^n \varepsilon (y(t_0) - t_0) + \varepsilon + t_n$$
$$= y(t_n) - \cancel{t_n} + (1+\varepsilon)^n \varepsilon (y(t_0) - t_0) + \varepsilon + \cancel{t_n}$$
$$= y(t_n) + (1+\varepsilon)^n \varepsilon (y(t_0) - t_0) + \varepsilon \quad \text{we'll use } *$$

Need to check $\varepsilon(y(t_n) - t_n) = (1+\varepsilon)^n \varepsilon (y(t_0) - t_0)$, which is true by assumption

d) fix $t > t_0$ and define $\varepsilon = (t - t_0)/n \Rightarrow t_n = t \;\forall n$ now consider the limit

$$\lim_{n\to\infty} y(t) = \lim_{n\to\infty} \left(1 + \frac{(t-t_0)}{n}\right)^n (y(t_0) - t_0) + t_n = (y_0 - t_0)\exp(t - t_0) + t$$

we use the fact that $\lim_{n\to\infty}\left(1 + \frac{a}{n}\right)^n = \exp(a)$

Quiz: Find an integrating factor for the non exact equation:

$$2\sin(y) + x\cos(y)\,y' = -1$$

orginal question was $2x\sin(y) + x^2\cos(y)\,y' = -1$

Ans: Rewrite: $\underbrace{(2\sin(y) + 1)}_{M}dx + \underbrace{x\cos(y)}_{N}dy = 0$

$$M_y = 2\cos(y), \quad N_x = \cos(y)$$

Thus we check $\dfrac{M_y - N_x}{N} = \dfrac{2\cos(y) - \cos(y)}{x\cos(y)} = \dfrac{1}{x}$ only a function of x!

$$\therefore \mu(x) = \exp\left(\int \frac{dx}{x}\right) = x$$

Or we could check $\dfrac{N_x - M_y}{M} = \dfrac{-\cos(y)}{2\sin(y) + 1}$ only a function of y

Remark! Both will work. This is an example of how μ is not uniqu

$$\therefore \mu(y) = \exp\left(\int \frac{-\cos(y)}{2\sin(y)+1}dy\right) = \exp\left(\int \frac{-du}{2u}\right) = \frac{1}{\sqrt{u}} \quad \text{where } u = 2\sin(y) + 1$$