Tutorial 10 - MAT244 - C.J. Adkins

Stability & Phase Planes

Phase Portraits, with more detail! $\dot{x} = Ax$ with $A \in M_{2\times2}(\mathbb{C})$, types of solutions:

① $\lambda_+, \lambda_- \in \mathbb{R}$ & $\mathrm{sgn}(\lambda_+) = \mathrm{sgn}(\lambda_-)$, as we saw before

$$\vec{x} = A\vec{\lambda}_+ \exp(\lambda_+ t) + B\vec{\lambda}_- \exp(\lambda_- t) \quad w/ \; A, B \in \mathbb{R}$$

$\boxed{\text{Node}}$

Suppose $\lambda_+ > \lambda_- > 0$ first, then $e^{\lambda_+ t}$ dominates $e^{\lambda_- t}$, i.e. $e^{\lambda_+ t} >> e^{\lambda_- t}$ for large t

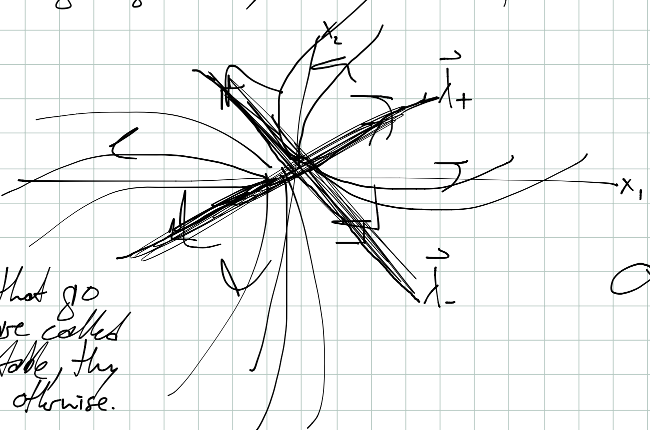
$$\Rightarrow \vec{x} = \exp(\lambda_+ t)\left[B\vec{\lambda}_+ + A\vec{\lambda}_+ \exp((\lambda_- - \lambda_+)t)\right]$$

notice this is negative

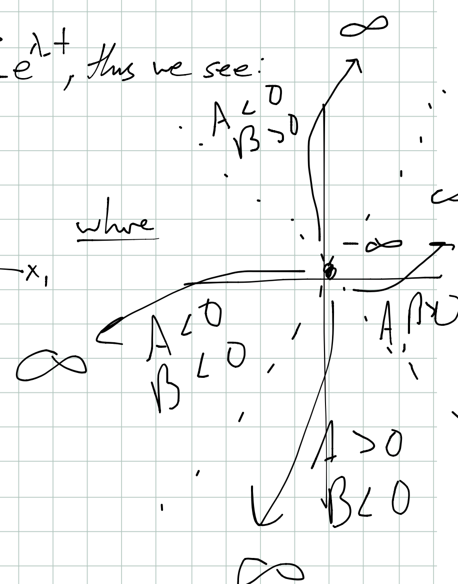
$\therefore \quad \vec{x}(t) \approx B\vec{\lambda}_+ e^{\lambda_+ t}$ for large t

by looking at large negative t, we see $\vec{x}(t) \approx A\vec{\lambda}_- e^{\lambda_- t}$, thus we see:

Unstable



where



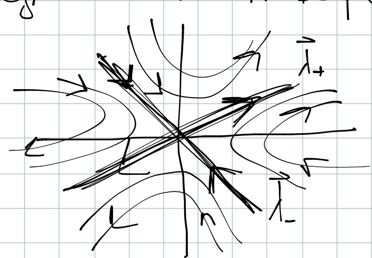
Remark: Things that go to the origin, are called asymptotically stable, they are called unstable otherwise.

② $\lambda_+, \lambda_- \in \mathbb{R}$ & $\mathrm{sgn}(\lambda_+) \neq \mathrm{sgn}(\lambda_-)$

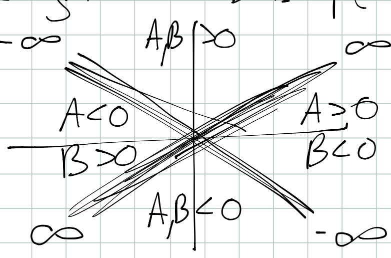
$\boxed{\text{Saddle}}$

$\Rightarrow x = A\vec{\lambda}_+ \exp(\lambda_+ t) + B\vec{\lambda}_- \exp(\lambda_- t)$, Notice that

large $t \Rightarrow x \approx A\vec{\lambda}_+ \exp(\lambda_+ t)$ (pos), large neg $t \Rightarrow x \approx B\vec{\lambda}_- \exp(\lambda_- t)$ (neg), thus



where



Always unstable

Ex (9.1 - #13) Find crit point and classify it and determine stability for:

$$\dot{x} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} x - \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Well, crit point $\iff \dot{x}=0 \implies \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \iff x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a crit point.

let $y = x - x_0$, $\implies \dot{y} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} y$ (which is centered around $\vec{0}$)

Eigenvalues? $P(\lambda) = (\lambda-1)(\lambda+1) - 1 = \lambda^2 - 2 \implies \lambda_+ = \sqrt{2}$, $\lambda_- = -\sqrt{2}$

By eigenvalues, we see this is a saddle crit point $\implies$ Unstable ■

Ex (9.1-#20) Consider

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{where } a_{ij} \in \mathbb{R}$$

let $p = a_{11} + a_{22} = \text{trace}(A)$, $q = a_{11}a_{22} - a_{12}a_{21} = \det(A)$ & $\Delta = p^2 - 4q$ (discriminate)

Show 0 is:

a) Node, if $q > 0$ & $\Delta \geq 0$

Well, it's easy to see $P(\lambda) = \lambda^2 - p\lambda + q$

$\implies P(\lambda) = 0 \iff \lambda_\pm = \dfrac{p \pm \sqrt{p^2 - 4q}}{2} = \dfrac{p \pm \sqrt{\Delta}}{2}$, thus if $q > 0 \implies \lambda \neq 0$ & if $\Delta \geq 0 \implies \text{sgn}(\lambda_+) = \text{sgn}(\lambda_-) = \text{sgn}(p)$

$\therefore$ we indeed have a node ■

b) Saddle, if $q < 0$

Well, $q < 0 \implies \Delta > p^2 \geq 0$ $\therefore$ $\lambda_\pm = \dfrac{p \pm \sqrt{\Delta}}{2} \implies \text{sgn}(\lambda_+) \neq \text{sgn}(\lambda_-)$

$\therefore$ we indeed have a saddle.

c) spiral, if $p \neq 0$ & $\Delta < 0$

complex root

Well, $p \neq 0$ & $\Delta < 0 \implies \lambda_\pm = \dfrac{p \pm i\sqrt{-\Delta}}{2} \implies$ spiral point ■

d) Center, if $p = 0$ & $q > 0$

Well, $q > 0$ & $p = 0 \implies \Delta < 0$, $\therefore$ $\lambda_\pm = i\sqrt{q} \implies$ center ■

Autonomous Systems, i.e

$$x' = F(x,y) \ \& \ y' = G(x,y) \ \text{w/} \ (x,y)(t_0) = (x_0, y_0), \ \text{i.e } \dot{x} = \vec{f}(x)$$

$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{f}(x) = \begin{pmatrix} F \\ G \end{pmatrix}$, crit points $\iff \vec{f}(x) = 0$ i.e $\dot{\vec{x}} = 0$

Trajectories of 2d · Autonomous System? $\dfrac{dy}{dx} = \dfrac{y'}{x'} = \dfrac{G}{F}$, this solution if possible are the paths

Ex(9.2-#4) Find trajectory for

$$x' = ay \ \& \ y' = -bx \quad , a,b>0 \quad , x(0)=\sqrt{a} \quad , y(0)=0$$

Well....

$$\frac{dy}{dx} = \frac{-bx}{ay} \iff \int y\,dy = -\int \frac{b}{a} x\,dx \iff y^2 = C - \frac{b}{a}x^2$$

Initial condition? $\Rightarrow \frac{y^2}{b} + \frac{x^2}{a} = 1 \Rightarrow$ trajectories are ellipses! Can check with old method:

The O.D.E system is

$$\dot{x} = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix} x \text{, check eigenvalues, } p(\lambda) = \lambda^2 + ab \Rightarrow \lambda_\pm = \pm i\sqrt{ab} \to \text{center}$$

Locally linear Systems:

$\dot{x} = Ax$ is linear, but autonomous systems have form $\dot{x} = f(x)$ (Not as nice!)

So lets Approximate the non-linear system: It'd be nice if

$\dot{x} = \vec{f}(x) = Ax + \vec{g}(x) \leftarrow$ "higher order terms" & $g(x)$ was small so $\dot{x} \approx Ax$.

This is the case if $\frac{|g(x)|}{|x|} \to 0$ as 0, i.e contains terms like x^n, $n > 1$

Ex(9.2-#4) Show 0 is a crit point & linearize, what type of system for

$$x' = x + y^2 \quad , y' = x + y$$

So... crit point $\iff \dot{x} = 0$, which is true, $x' = 0 + 0^2 = 0$, $y' = 0 + 0 = 0$ ✓

To linearize, notice $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^2 \\ 0 \end{pmatrix}$

thus if y is small (close to $\vec{0}$), we have $\dot{x} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x$

eigenvalues? we can read them of diagonal, $\boxed{\lambda_\pm = 1}$

but we only have 1 eigenvector :(, thus we need a generalized one.

$\therefore$ this is a unstable improper node. Since $\lambda = \lambda_-$ & missing a vector

Notice that $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is also a critical point: Thus the system about $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is:

$x \to x-1$
$y \to y+1$ i.e

$$x' = (x-1) + (y+1)^2 = x + 2y + y^2 \iff \dot{x} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} y^2 \\ 0 \end{pmatrix}$$
$$y' = (x-1) + (y+1) = x + y$$

Which shows the linearization as $\dot{x} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} x$

What are we doing here? first order 2-d Talyor expansion!

I.e. $\vec{f}(x) = \vec{f}(x_0) + \underset{\uparrow \text{Jacobian}}{J(x_0)}(x-x_0) + \frac{1}{2}\underset{\uparrow \text{Hessian}}{H(x_0)}(x-x_0)^2 + \cdots$ (should have seen this in multivariable calc)

If $x - x_0$ is small, then

$$\vec{f}(x) \approx \vec{f}(x_0) + J(x_0)(x-x_0)$$

By translation, we can recenter about $\vec{0}$, i.e. $\dot{x}_0 = \vec{f}(x_0) \overset{\text{crit point!}}{=} 0$, thus $\vec{y} = \vec{x} - \vec{x}_0$, $\dot{\vec{y}} = \dot{\vec{x}} - \dot{\vec{x}}_0$ gives

$$\dot{y} = J(x_0)\,y \text{, or } \begin{pmatrix} \dot{x} - F(x_0) \\ \dot{y} - G(y) \end{pmatrix} = \begin{pmatrix} \dot{x} - 0 \\ \dot{y} - 0 \end{pmatrix} = \boxed{\frac{d}{dt}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = J(x_0, y_0)\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}}$$

Where $J(x,y) = \begin{pmatrix} \partial_x F & \partial_y F \\ \partial_x G & \partial_y G \end{pmatrix}$

Notice in the previous example, $F(x,y) = x + y^2$, $G(x,y) = x + y$

$\Rightarrow J(x,y) = \begin{pmatrix} 1 & 2y \\ 1 & 1 \end{pmatrix}$, i.e $J(0,0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ & $J(-1,1) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ ← that's what we saw earlier!

So.... long story short, we just need to find crit points & J to linearize.

Ex(9.3-#3) Find crit points & linearize

$$x' = (1+x)\sin y \qquad y' = 1 - x - \cos y$$

Well, solve $\dot{x} = 0$, we see $(1+x)\sin y = 0$ if $\begin{cases} x = -1 \\ y = n\pi, \ n \in \mathbb{Z} \end{cases}$

$$x = -1 \Rightarrow 2 - \cos y = 0 \text{ (impossible!)}$$

$\therefore y = n\pi \Rightarrow 1 - x \mp 1$ (− if n even, + if n odd)

$\therefore (0, 2n\pi)$ & $(2, (2n+1)n\pi)$ are crit points

What is J? Well

$F = (1+x)\sin y$, $G = 1 - x - \cos y$ & $J = \begin{pmatrix} \partial_x F & \partial_y F \\ \partial_x G & \partial_y G \end{pmatrix} = \begin{pmatrix} \sin y & (1+x)\cos y \\ -1 & \sin y \end{pmatrix}$

$\therefore$ The linear system about $(0, 2n)$ $n \in \mathbb{Z}$ is

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$$

About $(2, (2n+1)n\pi)$ we have $\dot{x} = \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix} x$ ∎