MAT244 - Ordinary Differential Equations - Summer 2014
Assignment 3 Due: July 16, 2014

Full Name: ______________________________

Last First

Student #: ______________________________

Indicate which Tutorial Section you attend by filling in the appropriate circle:

◯	Tut 01	W 12:10-13:00	WI 523	Christopher Adkins
◯	Tut 02	W 17:10 - 16:00	LM 158	Yuri Cher
◯	Tut 03	W 17:10 - 16:00	SS 1074	Alexander Caviedes

Instructions:

- Due July 16, 2014 before the lecture at 13:10pm in MP203.
- Assignments should be completed individually.
- Write your solutions clearly, showing all steps. The solutions presented should not be your first draft! Grading is based on correctness as well as presentation.
- Assignments may be submitted up to one week after their due date, scanned, or carefully photographed, and submitted in a PDF via email to the course instructor: craig.sinnamon@utoronto.ca
- There will be a 5% reduction in mark per day of lateness beginning after 13:10pm on the day the assignment is due.
- Assignments may be submitted to the course instructor for remarking during office hours. Assignments that do not meet the following criterion may not be accepted for remarking.
  - the assignment was returned less than **EIGHT** days ago
  - the assignment is written in pen
  - the assignment is accompanied with an attached note clearly explaining the grading complaint
- Note that grades may decrease after remarking.

1. Find a particular solution to the nonhomogeneous equation

$$y''(t) - 2y'(t) + y(t) = \frac{e^t}{1+t^2} = g(t)$$

a) Hom-part

$y'' - 2y' + y = 0 \Rightarrow y_1 = e^t, \; y_2 = te^t$

b) Non-hom part, use variation of constants

$\therefore \; y(t) = uy_1 + vy_2$ where

$u = -\int \frac{gy_2}{W}, \quad v = \int \frac{gy_1}{W}$

Find Wronskian! $W = y_1y_2' - y_2y_1' = e^t(e^t + te^t) - te^t(e^t) = e^{2t}$

$\Rightarrow u = -\int \frac{t}{1+t^2}\,dt = \frac{1}{2}\ln|1+t^2| + A$

$v = \int \frac{dt}{1+t^2} = \arctan(t) + B$

This means the general solution is:

$y(t) = Ae^t + Bte^t + \frac{e^t}{2}\ln|1+t^2| + te^t\arctan(t)$

Pick any $A, B \in \mathbb{R}$ for a solution $\blacksquare$

2. a) Verify that $x(t) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}$ satisfies the differential equation

$$x'(t) = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x(t)$$

$RHS = x' = 2\begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t} = 2x$

$LHS = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t} = \begin{pmatrix} 12-4 \\ 8-4 \end{pmatrix} e^{2t} = 2\begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t} = 2x = R.H.S$

b) Verify that $\Psi(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$ satisfies the differential equation

$$\Psi'(t) = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \Psi(t)$$

$RHS = \psi' = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}$

$L.H.S = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix} = \begin{pmatrix} (1-4)e^{-3t} & (1+1)e^{2t} \\ (4+8)e^{-3t} & (4-2)e^{2t} \end{pmatrix} = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}$

3. Find the general solution of the system of equations

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} \mathbf{x}(t)$$

$x' = Ax$, we look for solution $x = e^{At}$

You can find that the eigenvalues & eigenvectors of A are:

$\lambda_1 = -2$, $\vec{\lambda}_1 = (-4, 5, 7)$

$\lambda_2 = 2$, $\lambda_2 = (0, -1, 1)$

$\lambda_3 = -1$, $\lambda_3 = (-3, 4, 2)$

$\therefore A = \Lambda^{-1} D \Lambda$ where $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ & $\Lambda = \begin{pmatrix} -4 & 0 & -3 \\ 5 & -1 & 4 \\ 7 & 1 & 2 \end{pmatrix}$

Thus $\vec{x} = A \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix} e^{-2t} + B \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t} + C \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} e^{-t}$

4. For what values of α is $\mathbf{x} = \mathbf{0}$ a spiral point.

$$\mathbf{x}'(t) = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \mathbf{x}(t)$$

Spiral point $\Leftrightarrow \lambda_1 = \bar{\lambda}_2$ & $\mathrm{Re}\,\lambda_1 \neq 0$ where λ_1 & λ_2 are eigenvalues of A. We have $\vec{x}' = A\vec{x}$

The eigenvalues are

$$\lambda_1 = \frac{1}{2}\left(\alpha - \sqrt{\alpha^2 - 20}\right) \quad \& \quad \lambda_2 = \frac{1}{2}\left(\alpha + \sqrt{\alpha^2 - 20}\right)$$

Since we need $\lambda_1 = \bar{\lambda}_2$ & $\mathrm{Re}(\lambda) \neq 0$

$\Rightarrow$ we need $\alpha^2 - 20 < 0$ & $\alpha \neq 0$ $\Leftrightarrow \alpha \in (-\sqrt{20}, 0) \cup (0, \sqrt{20})$

5. Find $A, B \in \mathbb{R}$ such that $\mathbf{x}(t) = \begin{pmatrix} 1 \\ A \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ B \end{pmatrix} te^{-3t}$ is a solution to the differential equation

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}(t)$$

L.H.S

Plug & Chug :)

$$x' = -3 \begin{pmatrix} 1 \\ A \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ B \end{pmatrix} e^{-3t} - 3 \begin{pmatrix} 1 \\ B \end{pmatrix} te^{-3t} = \begin{pmatrix} -2 \\ -3A+B \end{pmatrix} e^{-3t} - \begin{pmatrix} 3 \\ 3B \end{pmatrix} te^{-3t}$$

$$R.H.S = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \vec{x} = \begin{pmatrix} 1-4A \\ 4-7A \end{pmatrix} e^{-3t} + \begin{pmatrix} 1-4B \\ 4-7B \end{pmatrix} te^{t}$$, By comparison, we see

$1-4A = -2$ ①, $1-4B = -3$ ②, system overdetermined
$4-7A = -3A+B$ ③, $4-7B = -3B$ ④, fingers crossed!

$\boxed{A = 3/4}$, $\boxed{B = 1}$ by ① & ②

lets check ③ & ④, ④ is good, ③ is good ✓

$$\therefore \vec{x} = \begin{pmatrix} 1 \\ 3/4 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t}$$ works

6. Consider the following differential equation $y'' + 5y' + 4y = 0$, for constants $b, c \in \mathbb{R}$.

a) Determine a system of equations $x' = Ax$ that is equivalent to the differential equation.

$y''+py'+qy=0 \iff \vec{x}' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\vec{x}$ where $\vec{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$

It follows by expanding the system ∎

b) Suppose that y_1, y_2 form a fundamental set of solutions for the differential equation, and $x^{(1)}, x^{(2)}$ form a fundamental set of solutions for the equivalent system. Show that $W[y_1, y_2](t) = k\, W[x^{(1)}, x^{(2)}](t)$ for some $k \in \mathbb{R}$.

(**Hint.** You don't have to solve for y_1, y_2 or $x^{(1)}, x^{(2)}$, but you can if you want to)

let $x^{(1)} = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$, $x^{(2)} = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$

Since the systems are equivalent, y must be some linear combination of x_{11} & x_{21} (since $x_1 = y$).

$\therefore$ $y_1 = Ax_{11} + Bx_{21}$ w/ $A, B, C, D \in \mathbb{C}$

$y_2 = Cx_{11} + Dx_{21}$

Note that $y_1' = Ax_{12} + Bx_{22}$ since $x_2 = y'$.

$y_2' = Cx_{12} + Dx_{22}$

$\therefore$ We have $W[x^{(1)}, x^{(2)}] = x_{11}x_{22} - x_{12}x_{21}$

$\Rightarrow W[y_1, y_2] = y_1y_2' - y_2y_1' = (AD - CB)W[x^{(1)}, x^{(2)}]$ ∎

i.e $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$ & K = det of transformation of $x \to y$