MAT244 - Ordinary Differential Equations - Summer 2014
Assignment 2   Due: June 11, 2014

Full Name: ____

Last   First

Student #: ____

Indicate which Tutorial Section you attend by filling in the appropriate circle:

◯	Tut 01	W 12:10-13:00	WI 523	Christopher Adkins
◯	Tut 02	W 17:10 - 16:00	LM 158	Yuri Cher
◯	Tut 03	W 17:10 - 16:00	SS 1074	Alexander Caviedes

Instructions:

- Due June 11, 2014 before the lecture at 13:10pm in MP203.
- Assignments should be completed individually.
- Write your solutions clearly, showing all steps. The solutions presented should not be your first draft! Grading is based on correctness as well as presentation.
- Assignments may be submitted up to one week after their due date, scanned, or carefully photographed, and submitted in a PDF via email to the course instructor: craig.sinnamon@utoronto.ca
- There will be a 5% reduction in mark per day of lateness beginning after 13:10pm on the day the assignment is due.
- Assignments may be submitted to the course instructor for remarking during office hours. Assignments that do not meet the following criterion may not be accepted for remarking.
  - the assignment was returned less than **EIGHT** days ago
  - the assignment is written in pen
  - the assignment is accompanied with an attached note clearly explaining the grading complaint
- Note that grades may decrease after remarking.

1. a) State the Existence and Uniqueness Theorem for first order nonlinear ordinary differential equations.

if $\dot{y}=f(t,y)$ w/ $y(t_0)=y_0$, then such a y exists for $t\in(t_0-h,t_0+h)$, $h>0$ if:

f & $\frac{\partial f}{\partial y}$ are continuous in $(t,y)\in[\alpha,\beta]\times[\gamma,\delta]$ $(\alpha<t_0-h<t_0<t_0+h<\beta)$

b) What can you conclude about the following initial value problem from the above theorem?

$$\sqrt{\tan(t+y(t))} = y'(t), \quad y(\pi/7) = 0$$

By the above $f(t,y)=\sqrt{\tan(t+y)}$ $y(\frac{\pi}{7})=0$, we check cont!

$y(\frac{\pi}{7})=0 \Rightarrow f(\frac{\pi}{7},0)=\sqrt{\tan(\frac{\pi}{7})}$

tan is only defined continuously in $(-\frac{\pi}{2},\frac{\pi}{2})$, so $-\frac{\pi}{2}<t+y<\frac{\pi}{2}$

$\frac{\partial f}{\partial y}=\frac{\sec^2(t+y)}{2\sqrt{\tan(t+y)}}$, this is only continuous if $\sin(t+y)\neq 0$ or $\cos(t+y)\neq 0$

$\Rightarrow 0<t+y<\frac{\pi}{2}$

$\therefore$ By the above theorem we know there is a solution for $t\in(\frac{\pi}{7}-h,\frac{\pi}{7}+h)$, $h\approx$ small

2. Given $\phi_0(t) = 0$, compute the first three Picard iterates $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$ for the initial value problem

$$= f(t,y)$$

$$y'(t) = t^2 y - t, \quad y(0) = 0$$

Recall:

$$\phi_0 = y_0, \quad \phi_K = y_0 + \int_{t_0}^{t} f(s, \phi_{k-1}(s))\, ds$$

Thus:

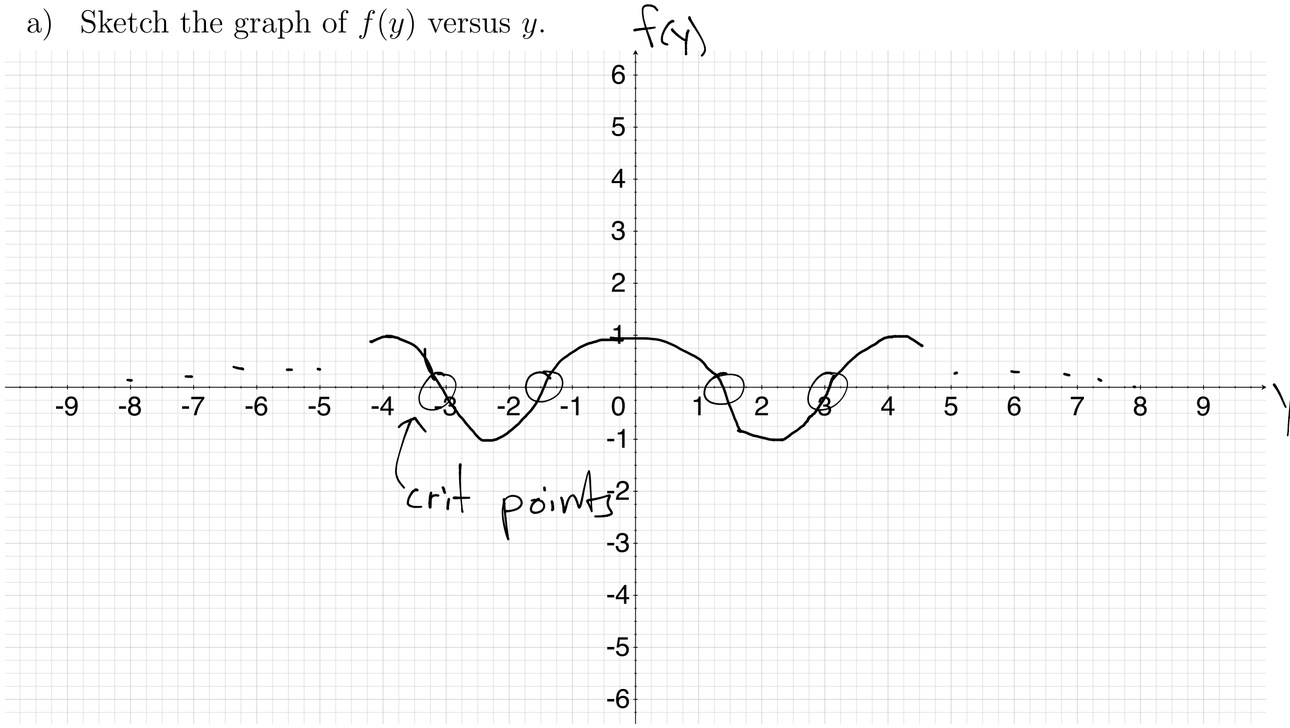
$$\phi_0 = 0, \quad \phi_1 = \int_0^t (0 - s)\, ds = -\frac{t^2}{2}$$

$$\phi_2 = \int_0^t s^2\left(-\frac{s^2}{2}\right) - s\, ds = -\int_0^t \frac{s^4}{2} + s\, ds = -\frac{t^5}{10} - \frac{t^2}{2}$$

$$\phi_3 = \int_0^t s^2\left(-\frac{s^5}{10} - \frac{s^2}{2}\right) - s\, ds = -\int_0^t \frac{s^7}{10} + \frac{s^4}{2} + s\, ds = -\frac{t^8}{80} - \frac{t^5}{10} - \frac{t^2}{2}$$

3. Consider the equation $\dfrac{dy}{dt} = f(y) = \cos(y)$.

a) Sketch the graph of $f(y)$ versus y.



b) Determine the critical (equilibrium) points, and classify each one as asymptotically stable, unstable, or semistable.

$y' = \cos(y) = 0 \iff y = \left(n + \frac{1}{2}\right)\pi \quad , n \in \mathbb{Z}$

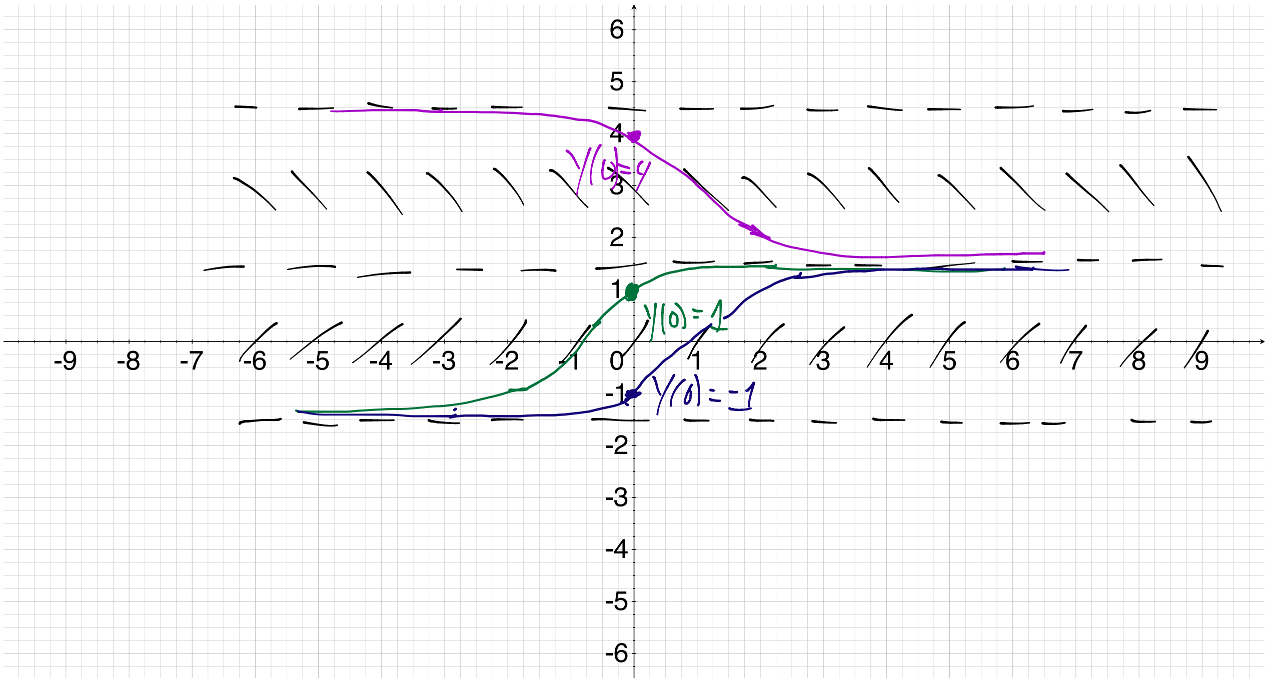
if n = odd #, eg $-3, -1, 1, 3, \ldots$ then $y'(n_-) < 0$, $y'(n_+) > 0$ $\Rightarrow$ unstable

if n = even #, eg. $-2, 0, 2, 4, \ldots$ then $y'(n_-) > 0$, $y'(n_+) < 0$, $\Rightarrow$ stable

c) On the set of axes below, sketch the graphs of the solutions to $\frac{dy}{dt} = f(y)$ with initial conditions:

$$y(0) = 1 \quad , \quad y(0) = 4 \quad , \text{ and } \quad y(0) = -1.$$

Note: Be sure to clearly label which initial condition each curve corresponds to.



d) Is there a solution $y = \phi(t)$ to $\frac{dy}{dt} = \sin(y)$ such that $\phi(0) < -1$ and $\phi(1) > 2$? Justify your answer.

No, since $\phi(t)=0 \Rightarrow$ crit point.

i.e solutions cannot pass through crit points!

4. Given $F(x, y(x)) = y\cos(xy)$, compute $\frac{d}{dx}F(x, y(x))$.

$F_x = y'\cos(xy) - y^2\sin(xy) - yxy'\sin(xy)$

5. If y is a function of x, solve the equation

$$-y\sin(xy) + 1 = xy'\sin(xy)$$

Notice that $x\sin(xy)\,dy + (y\sin(xy) - 1)\,dx) = 0$ is exact.

Since $F(x,y) = -\cos(xy) - x + C$ satisfies

$$dF(x,y) = x\sin(xy)\,dy + (y\sin(xy) - 1)\,dx = 0$$

$\Rightarrow y(x) = \dfrac{\arccos(x + \tilde{C})}{x}$ solves the O.D.E

6. Use Euler's method to find an approximate value of the solution to the following initial value problem at $t = 0.3$, which $h = 0.1$,

$$y'(t) = 2y(t) + t^2, \quad y(0) = 1$$

Recall: $y(x_{n+1}) = y(x_n) + h\,y'(x_n) = y(x_n) + \frac{2y(x_n) + x_n^2}{10}$

$\therefore$ $y(0) = 1$

$y(0.1) = 1 + \frac{2}{10} = \frac{6}{5}$

$y(0.2) = \frac{6}{5} + \frac{\frac{12}{5} + \frac{1}{10^2}}{10} = \frac{6}{5} + \frac{1}{4} = \frac{29}{20}$

$y(0.3) = \frac{29}{20} + \frac{\frac{29}{10} + \frac{1}{25}}{10} = \frac{29}{20} + \frac{29}{100} + \frac{1}{250} = \frac{164}{100} + \frac{1}{250}$