MAT237 - Tutorial 17 - 28 July 2015

1 Coverage

Vector derivatives, line integrals of scalar functions and vector fields, Green's theorem.

2 Problems

I suggest the following problems. Like last time, I won't have too much to comment about any of these, since they're mostly computational.

In retrospect, I sort of regret putting the third part of the first question on here, but it was so annoying to type out that I'm keeping it here for posterity's sake.

1. (BL 13.1.5) Let (x, y) be the usual Cartesian coordinates in $\mathbb{R}^2$. Recall that in polar coordinates, these can be written as $x = r\cos\theta$ and $y = r\sin\theta$.

   (a) Determine the standard polar unit vectors e_r and e_θ.

   (b) Using the multivariable chain rule, determine the ∇ operator in polar coordinates.

   (c) Similarly, obtain the Laplacian in polar coordinates.

2. (BL 13.2.5) Let $\mathbf{F}(x, y, z) = (x, y, z^2)$ and let C be the curve defined by the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = x$. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{x}$.

3. (BL 13.3.3)

   (a) Show that the vector field $\mathbf{F}(x, y, z) = (y\cos(xy), x\cos(xy) + e^z, ye^z)$ is conservative by finding its potential function.

   (b) Let C be the part of the curve given by the intersection of the sphere of radius 2 centred at the origin and the yz-plane above the x-axis. Orient C so that its tangent vectors point in the positive y direction. Compute $\oint_C \mathbf{F} \cdot d\mathbf{x}$.

4. (Parts of BL 13.3.5, with the statement of (a) corrected.)

   (a) Let $D \subseteq \mathbb{R}^2$ be a region whose boundary $C := \partial D$ is a piecewise-smooth simple closed curve. Show that the area $A(D)$ of D is given by:

   $$A(D) = \oint_C x\,dy = -\oint_C y\,dx = \oint_C \frac{1}{2}(x\,dy - y\,dx).$$

   (b) Use the formula above to compute the areas of some simple sorts of shapes, like circles, ellipses, or whatever else you can parametrize the boundary of. Note, for example, that this allows you to easily find the area of any polygon once you know its vertices.

3 Solutions and Comments

1. ***Solution***: Recall that the transformations the other way are $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(\frac{y}{x})$.

   (a) The unit vector $\hat{r}$ in the r-direction is easy to write down, since it's the unit length vector in the same direction as $(r\cos(\theta), r\sin(\theta))$, which is just $\hat{r} = (\cos(\theta), \sin(\theta))$. In the θ direction, we need the unit-length vector that is perpendicular to $\hat{r}$ pointing in the counterclockwise direction (measuring from the x-axis). We can just see that this is $\hat{\theta} = (-\sin(\theta), \cos(\theta))$.

   (b) As the question says, we use the chain rule. First, the calculation for $\frac{\partial}{\partial x}$:

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} - \frac{y}{x^2+y^2}\frac{\partial}{\partial \theta} = \cos(\theta)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta}.$$

   Similarly we compute $\frac{\partial}{\partial y}$ and end up with:

$$\frac{\partial}{\partial y} = \sin(\theta)\frac{\partial}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}.$$

   Combining these, we get:

$$\begin{aligned}\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) &= \left(\cos(\theta)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta}, \sin(\theta)\frac{\partial}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta}\right) \\ &= \hat{r}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial}{\partial \theta}.\end{aligned}$$

   (c) This is a bit annoying, but is an elementary application of the above. We compute:

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial x}\left(\cos(\theta)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right) \\ &= \cos(\theta)\frac{\partial}{\partial r}\left(\cos(\theta)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right) - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\left(\cos(\theta)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right) \\ &= \cos(\theta)\left(\cos(\theta)\frac{\partial^2}{\partial r^2} + \frac{\sin(\theta)}{r^2}\frac{\partial}{\partial \theta} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial r \partial \theta}\right) \\ &\qquad - \frac{\sin(\theta)}{r}\left(-\sin(\theta)\frac{\partial}{\partial r} + \cos(\theta)\frac{\partial}{\partial r\partial \theta} - \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta} - \frac{\sin(\theta)}{r}\frac{\partial^2}{\partial \theta^2}\right) \\ &= 2\frac{\sin(\theta)\cos(\theta)}{r^2}\frac{\partial}{\partial \theta} + \cos^2(\theta)\frac{\partial^2}{\partial r^2} - 2\frac{\sin(\theta)\cos(\theta)}{r}\frac{\partial}{\partial r\partial \theta} + \frac{\sin^2(\theta)}{r}\frac{\partial}{\partial r} + \frac{\sin^2(\theta)}{r^2}\frac{\partial^2}{\partial \theta^2}\end{aligned}$$

   The same sort of calculation yields:

$$\frac{\partial^2}{\partial y^2} = -2\frac{\sin(\theta)\cos(\theta)}{r^2}\frac{\partial}{\partial \theta} + \sin^2(\theta)\frac{\partial^2}{\partial r^2} + 2\frac{\sin(\theta)\cos(\theta)}{r}\frac{\partial}{\partial r\partial \theta} + \frac{\cos^2(\theta)}{r}\frac{\partial}{\partial r} + \frac{\cos^2(\theta)}{r^2}\frac{\partial^2}{\partial \theta^2}$$

   Adding these up, cancelling and using the Pythagorean identity yields:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

2. ***Solution***: Our vector field $\mathbf{F}$ is ∇f for $f(x,y,z) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^3}{3}$, meaning $\mathbf{F}$ is conservative. The curve C is a closed loop, and so the integral of $\mathbf{F}$ along the curve is 0.

   If we didn't know this fact, however, we could easily get the same result directly. A parametrization of the curve is given by $\gamma(t) = (\cos(t), \sin(t), \cos(t))$, for $t \in [0, 2\pi)$. Then $\gamma'(t) = (-\sin(t), \cos(t), -\sin(t))$, and we can compute:

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{x} &= \int_0^{2\pi} (\cos(t), \sin(t), \cos^2(t)) \cdot (-\sin(t), \cos(t), -\sin(t))\, dt \\ &= -\int_0^{2\pi} \cos^2(t)\sin(t)\, dt \\ &= 0\end{aligned}$$

3. ***Solution***: (a) We're looking for a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $f_x = y\cos(xy)$, $f_y = x\cos(xy) + e^z$ and $f_z = ye^z$.

   We start by integrating the first component of $\mathbf{F}$ with respect to x, obtaining $\sin(xy) + g(y,z)$ for some function g.

   Differentiating this with respect to y, we obtain $x\cos(xy) + g_y(x,y)$, which we'd like to equal $x\cos(xy) + e^z$, suggesting that we should let $g(y,z) = ye^z + h(z)$ for some function h.

   Finally, differentiating $\sin(xy) + ye^z + h(z)$ with respect to z, we obtain $ye^z + h'(z)$, which we'd like to equal ye^z, suggesting that h should be a constant function. We might as well set use the constant zero function, leaving us with

$$f(x,y,z) = \sin(xy) + ye^z$$

   as our potential function. One can easily verify that $\mathbf{F} = \nabla f$.

   (b) Since we've just shown that $\mathbf{F}$ is conservative, we know that this line integral depends only on the endpoints of C. These endpoints are $(0,-1,0)$ and $(0,1,0)$, and given the orientation in the question, we are traveling from the first point to the second point.

   So, by the fundamental theorem of calculus:

$$\oint_C \mathbf{F} \cdot d\mathbf{x} = f(0,1,0) - f(0,-1,0) = 1 - (-1) = 2.$$

4. (a) We know that the area of a region D is given by $\int\int_D dA$. To apply Green's Theorem to this problem, we first need to find a vector field $\mathbf{F} = (F_1, F_2)$ such that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$. There are many possible candidates for such a vector field. Three that come to mind are $\mathbf{F}(x,y) = (0,x)$, $\mathbf{F}(x,y) = (-y,0)$ and $\mathbf{F}(x,y) = \frac{1}{2}(-y,x)$. Let $\mathbf{F}$ be any one of these.

   Then applying Green's Theorem to D and its boundary (oriented positively), we have:

$$A(D) = \int\int_D dA = \int\int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA = \oint_C \mathbf{F} \cdot d\mathbf{x}.$$

All that remains is to note that the three possible **F**'s we listed above give the three outcomes in the problem.

(b) Start with a circle of radius R. This can be parametrized by $\gamma(t) = (R\cos(t), R\sin(t))$, for $t \in [0, 2\pi)$. Then we have $\gamma'(t) = (-R\sin(t), R\cos(t))$. For this first example, we'll test all the equivalent formulations from the problem:

$$\oint_C x\,dy = R^2 \int_0^{2\pi} \cos^2(t)\,dt = \pi\mathbb{R}^2,$$

$$-\oint_C y\,dx = -R^2 \int_0^{2\pi} -\sin^2(t)\,dt = \pi\mathbb{R}^2,$$

$$\oint_C \frac{1}{2}(x\,dy - y\,dx) = \frac{1}{2}\int_0^{2\pi} R^2\cos^2(t) + R^2\sin^2(t)\,dt = \frac{R^2}{2}\int_0^{2\pi} dt = \pi R^2$$

Now, an ellipse. A general ellipse C centred at the origin is given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is most naturally parametrized in a similar way to the circle, with $(a\cos(t), b\sin(t))$ for $t \in [0, 2\pi)$. We can then compute the area of the ellipse as:

$$\oint_C x\,dy = ab\int_0^{2\pi} \cos^2(t)\,dt = ab\pi.$$