Assignment 3

APM462 – Nonlinear Optimization – Summer 2016
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SOLUTIONS

Question 1 Let $x_1, \ldots, x_n$ and $r_1, \ldots, r_n$ be positive real numbers such that $\sum_{i=1}^n r_i = 1$. Prove that

$$\prod_{i=1}^n x_i^{r_i} \leqslant \sum_{i=1}^n r_i x_i$$

Solution Since $\varphi(x) = -\log x$ is a convex function and $\sum r_i = 1$ we have

$$\varphi\left(\sum_{i=1}^n r_i x_i\right) \leqslant \sum_{i=1}^n r_i \varphi(x_i) \implies \log\left(\prod_{i=1}^n x_i^{r_i}\right) \leqslant \log\left(\sum_{i=1}^n r_i x_i\right)$$

Since $\log x$ is monotonically increasing, we have that

$$\prod_{i=1}^n x_i^{r_i} \leqslant \sum_{i=1}^n r_i x_i$$

as desired.

Remark: For $n = 2$, convexity of φ implies $\varphi\left(\sum_{i=1}^n r_i x_i\right) \leqslant \sum_{i=1}^n r_i \varphi(x_i)$ when $\sum_{i=1}^n r_i = 1$. This remains true for integers $n \geqslant 2$. Prove it. $\square$

Question 2 Minimize the function $f(x, y) = -x - y$, subject to the constraints:

$$\begin{aligned} x^2 + y^2 &\leqslant 1 \\ (x-1)^2 + y^2 &\leqslant 1 \\ x &> -1 \\ y &> -1 \end{aligned}$$

(a) Find the point(s) which satisfy the 1st order condition.

(b) Check the 2nd order conditions for a minimum.

(c) What is the global minimum?

Solution

(a) Let

$$g_1(x,y) = x^2 + y^2 - 1 \quad \& \quad g_2(x,y) = (x-1)^2 + y^2 - 1$$

We see that that all points are regular except for $(0,0)$,$(1,0)$, and the first order conditions (only apply to the regular points) are given by

$$\nabla f + \mu_1 \nabla g_1 + \mu_2 \nabla g_2 = 0 \implies \begin{cases} -1 + 2\mu_1 x + 2\mu_2(x-1) = 0 \\ -1 + 2\mu_1 y + 2\mu_2 y = 0 \end{cases}$$

$$\mu_1, \mu_2 \geqslant 0; \quad \mu_1 g_1(x,y) = 0; \quad \mu_2 g_2(x,y) = 0.$$

Case I : $\mu_1 = 0, \mu_2 = 0$: This implies $\nabla f = 0$, a contradiction. So no solutions in this case.
Case II : $\mu_1 = 0, \mu_2 > 0$: Check no solutions.
Case III: $\mu_1 > 0, \mu_2 = 0$: $x = y = \mu_1 = \frac{1}{\sqrt{2}}$.
Case IV : $\mu_1 > 0, \mu_2 > 0$: Check no solutions.

So we get 3 points which are candidates for a local minimum: $(0,1),(1,0)$ (non-regular points) and $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ (regular point).

□

(b) The second order condition (for the regular point in case III) is given by

$$\nabla^2(f + \mu_1 g_1 + 0 g_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is positive definite. So the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a local minimum. □

(c) Since f is a continuous function and the domain given by the constraints is a compact set, we know there is a global minimum. Any global minimum is also a local minimum, so the global minimum is among our 3 candidates. We check the value of f at these 3 points: $f(0,0) = 0, f(1,0) = -1, f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = -\sqrt{2}$. So the global minimum value is $-\sqrt{2}$ which occures at the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. □

Question 3 Assume that f is a strictly convex function on $\mathbb{R}^2$, and that f attains its global minimum at the point $(-5,2)$ and nowhere else. For this function f, consider the minimization problem

$$\text{minimize } f \text{ in the set } \{(x,y) \in \mathbb{R}^2 : x \geqslant 0 \text{ and } y \geqslant 0\}$$

Prove that the minimum (x_*, y_*) must satisfy $x_* = 0$.

Solution Without the loss of generality, we may assume that $f(-5,2) = 0$. Now, for the sake of contradiction, suppose that the minimum occurs with $x_* > 0$. Then along the line from $\mathbf{y}_* = (-5,2)$ to this minimizer $\mathbf{x}_*$, we have (using the fact f is strictly convex)

$$f((1-t)\mathbf{y}_* + t\mathbf{x}_*) < t f(\mathbf{x}_*)$$

Take

$$t_* = \frac{5}{x_* + 5}$$

so we see

$$f(0, y(t_*)) < \frac{5}{x_* + 5} f(\mathbf{x}_*) < f(\mathbf{x}_*)$$

thus $\mathbf{x}_*$ isn't the true minimum, contradiction. Thus the minimum must occur on $x_* = 0$. □

Question 4 Fix $\alpha_i \in \mathbb{R}$ and consider the minimization problem

$$\begin{aligned} \text{minimize} \quad & f(x) = -\sum_{i=1}^{n} \log(\alpha_i + x_i) \\ \text{subject to} \quad & x_1, \ldots, x_n \geqslant 0 \\ & x_1 + \ldots + x_n = 1 \end{aligned}$$

(a) Using the 1st order conditions show that x_i has the form:

$$x_i = \max\{0, \frac{1}{\lambda} - \alpha_i\}$$

for some $\lambda \in \mathbb{R}$

(b) Using the equality constraint $x_1 + \ldots + x_n = 1$, argue that λ is unique, hence the solution to the minimization problem is unique. (You are not asked to solve for λ)

Solution

(a) Let $h(\mathbf{x}) = x_1 + \ldots + x_n - 1 = 0$ and $g_i(\mathbf{x}) = -x_i \leqslant 0$. Note that $\alpha_i + x_i > 0$ since otherwise log is undefined.

We have $(\nabla f)_i = \frac{-1}{\alpha_i + x_i}$, $(\nabla h)_i = 1$, and $(\nabla g_j)_i = -\delta_{ij}$. So the first order neccessary conditions are given by:

$$\nabla f + \lambda \nabla h + \sum \mu_j \nabla g_j = 0 \implies \frac{-1}{\alpha_i + x_i} + \lambda - \mu_i = 0$$
$$\mu_i x_i = 0, \quad \mu_i \geqslant 0, \quad x_1 + \cdots + x_n = 1$$

Case $x_i > 0$: $\mu_i = 0 \implies \lambda = \frac{1}{\alpha_i + x_i} \implies x_i = \frac{1}{\lambda} - \alpha_i > 0$.
Case $x_i = 0$: $\alpha_i > 0$. Since $\frac{-1}{\alpha_i} + \lambda - \mu_i = 0$, we get that $\frac{-1}{\alpha_i} + \lambda \geqslant 0$ and so $\frac{1}{\lambda} - \alpha_i \leqslant 0$.
It follows that $x_i = \max\{0, \frac{1}{\lambda} - \alpha_i\}$.

□

(b) Note that $x_i(\lambda) = \max\{0, \frac{1}{\lambda} - \alpha_i\}$ is a continuous, decreasing function of λ. So $x(\lambda) := x_1(\lambda) + \ldots + x_n(\lambda)$ is also a continuous, decreasing function of λ. For $\lambda > 0$ small, $x_i(\lambda)$ is large, and as λ is getting larger, $x_i(\lambda)$ eventualy becomes 0. Also $x(\lambda)$ behaves in the same way and since it is continuous, there will be some λ for which $x(\lambda) = 1$.

Note that at for least one i the $x_i(\lambda) > 0$ since otherwise $x(\lambda)$ would never equal 1. So for at least one i, $x_i(\lambda) = \frac{1}{\lambda} - \alpha_i$ which is *strictly* decreasing. Hence near the value 1 the function $x(\lambda)$ is strictly decreasing. So the solution must be unique.

□

Question 5 Consider the minimization problem

$$\begin{aligned} \text{minimize} \quad & f(x,y) = -y \\ \text{subject to} \quad & x^2 + y^2 - 1 \geqslant 0 \\ & x^2 + (y+1)^2 - 4 \leqslant 0 \end{aligned}$$

(a) Explain (in one sentence) why there is a solution to this problem.

(b) Find the candidates for the minimizers, i.e. points satisfying the 1st order conditions (careful about which points are regular).

(c) Check the 2nd order conditions to decide if any of the candidates is are minimizing points.

(d) Find the minimizing point.

Solution

(a) It's easy to see the constraints restrict f (continuous) to a compact domain, thus it'll have a solution.

(b) Let

$$g_1(x,y) = -x^2 - y^2 + 1 \quad \& \quad g_2(x,y) = x^2 + (y+1)^2 - 4$$

Then all points except $(0,1)$ are regular. The first order condition (for regular points) is given by

$$\nabla f + \mu_1 \nabla g_1 + \mu_2 \nabla g_2 = 0 \implies \begin{cases} 0 - 2\mu_1 x + 2\mu_2 x = 0 \\ -1 - 2\mu_1 y + 2\mu_2 (y+1) = 0 \end{cases}$$

$$\mu_1, \mu_2 \geqslant 0; \quad \mu_1 g_1(x,y) = 0; \quad \mu_2 g_2(x,y) = 0.$$

Case I : $\mu_1 = 0, \mu_2 = 0$: This implies $\nabla f = 0$, a contradiction. So no solutions in this case.
Case II : $\mu_1 = 0, \mu_2 > 0$: $x = 0, y = 1, \mu_2 = \frac{1}{4}$.
Case III: $\mu_1 > 0, \mu_2 = 0$: $x = 0, y = -1, \mu_1 = \frac{1}{2}$.
Case IV : $\mu_1 > 0, \mu_2 > 0$: $x = 0, y = 1$.

So we get 2 points which are candidates for a local minimum: $(0,1)$ (non-regular points) and $(0,-1)$ (regular point).

□

(c) The second order condition (for the regular point in case III) is given by

$$\nabla^2(f + \mu_1 g_1 + 0 g_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is negative definite. So the point $(0,-1)$ is not a local minimum.

□

(d) Since f is a continuous function and the domain given by the constraints is a compact set, we know there is a global minimum. Any global minimum is also a local minimum, so the global minimum point is among our 2 candidates, one of which is not a local minimum. So the global minimum point must be the non-regular point $(0, 1)$. Note: this can also be seen by considering the level sets of the function f.

□